FUNCTORALITY OF THE GAMMA FILTRATION AND COMPUTATIONS FOR SOME TWISTED FLAG VARIETIES

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Abstract. We introduce techniques for uniformly studying the gamma filtration of projective homogeneous varieties. These techniques are utilized in some cases of inner-twisted flag varieties (of type A) to show that functorality known for the Chow rings of these varieties also extends to the associated graded rings for the gamma filtrations of the same varieties. As an application, we show that the associated graded groups for the gamma filtration of these varieties are torsion free in low homological degrees.

1. introduction

In the past, the γ-filtration, along with the coniveau – or topological, or Chow – filtration, of the Grothendieck ring of a projective homogeneous variety X had been studied in order to gain information on the, often more elusive, Chow groups of X. For example, these filtrations facilitated the first calculation of torsion in the Chow groups of a projective quadric [Kar90] and the first computation of torsion in the Chow group of codimension 2 cycles of a Severi-Brauer variety [Kar95b]. In return, knowledge about the Chow groups of these varieties often led to information about these varieties, or related objects, including the construction of fields of u-invariant 9 [Izh01] or showing generic central simple algebras of prime exponent were indecomposable [Kar95b].

More recently, the γ-filtration has been used to estimate the size of torsion in the Chow groups of higher codimension for Severi-Brauer varieties [Bae15] and to estimate torsion in the Chow groups of codimension 2 cycles and codimension 3 cycles for many other projective homogeneous varieties [GZ14]. The γ-filtration has also recently been shown to have connections to the theory of cohomological invariants [MNZ15] due to the relations between the γ-filtration and the Chow group of codimension 2 cycles, and the Chow group of codimension 2 cycles for generic complete flag varieties and cohomological invariants of degree 3.

Very recently, Karpenko conjectured that the γ-filtration should completely compute the Chow ring for the class of generically split generic twisted flag varieties. More precisely, the Chow ring of such a variety X is generated by Chern classes [Kar18c]. This means that the γ-filtration and coniveau filtration for this X coincide and Karpenko's conjecture is that the canonical epimorphism from the Chow ring of X to the associated graded ring for the coniveau filtration of X is an isomorphism. It's now known that this conjecture is false in general [Kar19] but, it has been proved in a number of cases [Kar17b, Kar18b, Kar18a, KM19] and is still open in many more.

This paper is the result of studying the associated graded ring for the γ-filtration of an arbitrary Severi-Brauer variety (it is still open whether or not Karpenko's conjecture holds in this case; see [KM19] for partial results in this direction). We prove two main theorems in this regard: Theorem 4.11 and Theorem 5.1. The first of these theorems extends functorality that is known to hold for the Chow ring (and to the associated graded ring for the coniveau filtration) of a Severi-Brauer variety to functorality for the associated graded ring for the γ-filtration. The second of these theorems is

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a direct computation of the group summands of the associated graded ring for the γ-filtration in low homological degrees; in particular, these summands are torsion free. Throughout this paper we also show how to extend these theorems to other (generically split) inner twisted flag varieties of type A.

It seems that, although all of our results are stated and worked out only in the case of Severi-Brauer and related varieties, the ideas contained here should extend to other classes of projective homogeneous varieties. For this reason we're going to spend some time explaining the aspects that should generalize to other settings.

The first aspect of this paper that should be explained is the use of τ-functorial replacements for a smooth variety X (Definition 4.1). Essentially, a τ-functorial replacement for X is a smooth variety Y that has the two properties: the associated graded rings, $\mathrm{gr}_\gamma K(X)$ and $\mathrm{gr}_\gamma K(Y)$, for the γ-filtrations of X and Y are isomorphic; the associated graded rings for the γ-filtration and coniveau filtration, $\mathrm{gr}_\gamma K(Y)$ and $\mathrm{gr}_\tau G(Y)$, of Y are canonically isomorphic. In particular, the ring $\mathrm{gr}_\gamma K(Y)$ computes the ring $\mathrm{gr}_\gamma K(X)$ and $\mathrm{gr}_\gamma K(Y)$ has all of the functorality of $\mathrm{gr}_\tau G(Y)$.

To the author's knowledge, the first time τ-functorial replacements appeared in the literature is in [Kar98] where they were used to compute the torsion subgroup of the Chow group of codimension 2 cycles of a Severi-Brauer variety in some generic cases. Here we use τ-functorial replacements to prove functorality results for the γ-filtration and its associated graded ring. As an example, see Corollary 4.12, we show that the associated graded ring for the γ-filtration of a Severi-Brauer variety X is a sum of copies of the associated graded ring for the γ-filtration of the minimal Severi-Brauer variety X' Brauer-equivalent to X.

Similarly, the general philosophy working with τ-functorial replacements should be: if one can obtain a decomposition of the motive of X depending only on some canonically associated subvarieties of, and projective bundles over, X and, if X has a τ-functorial replacement Y that also has this decomposition, then the ring $\mathrm{gr}_\gamma K(X)$ should decompose similarly. The reason for this is because the coniveau filtration already has most of the functorality one needs to make this claim, e.g. it has pushforwards. So, if Y is a τ-functorial replacement for X and if $f : Z \to Y$ is a proper morphism then it makes sense to talk about the pushforward $f_* : \mathrm{gr}_\gamma K(Z) \to \mathrm{gr}_\gamma K(Y)$ defined as the uniquely determined arrow making the following square commutative.

$$\begin{array}{ccc} \mathrm{gr}_\gamma K(Z) & \longrightarrow & \mathrm{gr}_\tau G(Z) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ \mathrm{gr}_\gamma K(Y) & = & \mathrm{gr}_\tau G(Y) \end{array}$$

And this should be sufficient to make the claim for $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\gamma K(X)$.

The second aspect of this paper that should be explained is the method we use for calculating the γ-filtration of a Severi-Brauer variety. For any Severi-Brauer variety X associated to a p-primary indexed central simple algebra A, one can find a finite set of λ-ring generators for the Grothendieck ring $K(X)$. The finite set that we use is a collection of sheaves, or vector bundles, that comes from the data contained in the reduced behavior of A (Definition 5.2 and Lemma 5.3).

It follows almost immediately that any set of λ-ring generators for $K(X)$, with X a smooth variety, also determines generators for the γ-filtration of X: one can take as generators for γ^i those K-theoretic Chern classes of the λ-ring generating set (below we take the negatives of the duals of this set, since this is more convenient for computations). A possibly naïve, but still interesting, question would be whether this idea extends to other projective homogeneous varieties: is there a canonical set of sheaves, or vector bundles, associated to a discrete invariant of a projective homogeneous variety X that generates $K(X)$ as a λ-ring? As one could just take a basis for $K(X)$ for their λ-ring generating set, a better but, more difficult, question is: is there a canonical set S_X

of sheaves, or vector bundles, associated to a discrete invariant of a projective homogeneous variety X such that S_X generates $K(X)$ as a λ-ring *and* S_X is minimal among such sets?

Now to an overview of this paper. Section 2 and section 3 serves as background to section 4 and section 5. In Section 2 we describe a nice presentation for the Grothendieck ring of a Severi-Brauer variety. Section 3 gives the definition of the γ and coniveau filtrations; we take the properties of these filtrations as known and refer to references when the reader needs them. Section 4 contains the main bulk of conceptual work. In this section we introduce τ-functorial replacements and prove that they exist in a number of cases. Section 5 is, by contrast, mainly computational. We compute here the associated graded ring for the γ-filtration of a Severi-Brauer variety through entirely elementary means.

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise.

If X is a variety considered over a field F, not necessarily equal to k, we write $\overline{X}$ for X over an algebraic closure of F.

If p is a prime, then v_p is the p-adic valuation.

2. Grothendieck groups of Severi-Brauer varieties

Throughout this section we fix a central simple algebra A of degree n and let

$$X = \mathrm{SB}(A) \subset \mathrm{Gr}(n, A)$$

be the Severi-Brauer variety of A of dimension $n-1$ considered as a subvariety of the Grassmannian of n-planes in A. For any field F over k, the F-points of $\mathrm{SB}(A)$ are exactly the minimal right ideals of A_F. We write ζ_X for the tautological sheaf on X. By definition, ζ_X is the pullback of the universal subsheaf on $\mathrm{Gr}(n, A)$ so, for any k-algebra R and any R-point x of X corresponding to a right ideal $I \subset A \otimes_k R$, the sheaf $x^*\zeta_X$ can be canonically identified with I when considered as an R-module; in particular, ζ_X is a right module over the constant sheaf A.

By $K(X)$ we mean the Grothendieck ring of locally free sheaves on X. By $G(X)$ we mean the Grothendieck ring of coherent sheaves on X. The two groups are canonically isomorphic via the morphism sending the class of a locally free sheaf in $K(X)$ to the class of itself in $G(X)$. These groups have been computed in this case:

Theorem 2.1 ([Qui73, §8, Theorem 4.1])**.** *The homomorphism of K-groups*

$$\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X)$$

sending the class of a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

In particular, $K(X)$ is a free $\mathbb{Z}$-module of rank $\deg(A)$ that is additively generated by the classes

$$\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$$

as i varies between $0 \leq i < \deg(A)$; here we denote by M_i a simple $A^{\otimes i}$-module. For any splitting field F of A, the variety X_F is isomorphic with the projective space $\mathbb{P}^n_F$, the extension of scalars map $K(X) \to K(X_F)$ is injective, and identifies $K(X)$ as a subring of $K(X_F)$. More precisely, we have:

Theorem 2.2. *In the setting above, let ξ denote the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$. There is a ring isomorphism*

$$\mathbb{Z}[x]/(1-x)^n \xrightarrow{\sim} K(X_F)$$

sending x to ξ.

Under this isomorphism $K(X)$ identifies with the subring of $\mathbb{Z}[x]/(1-x)^n$ generated by $\mathrm{ind}(A^{\otimes i})x^i$.

Proof. The isomorphism is well-known, see [Man69]. Finally, we use that $\zeta_X \otimes_k F$ has class $\deg(A)\xi$ in $K(X_F)$ to get the remaining claim by computing the ranks of the $\zeta_X(i)$. □

We also include here the following formulas. The first is just the binomial theorem (before and after a change of coordinates). The second applies the previous one.

Lemma 2.3. *In any commutative ring there are equalities, for any integers $n \geq i \geq 0$,*

$$(1) \qquad (1-x)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i \quad \textit{and} \quad x^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (1-x)^i$$

$$(2) \qquad x^n - 1 = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (1-x)^i.$$

□

3. THE GAMMA AND CONIVEAU FILTRATIONS

In this section we recall some results on the γ-filtration of $K(X)$ and on the coniveau (or topological or Chow) filtration on $G(X)$ for an arbitrary smooth variety X.

For the first, recall there are γ-operations defined on $K(X)$ as follows. The ith-exterior power operation induces a well-defined map $\lambda^i : K(X) \to K(X)$ which is uniquely determined by sending the class of a locally free sheaf $\mathcal{F}$ to the class of $\wedge^i \mathcal{F}$. The ith γ operation $\gamma^i : K(X) \to K(X)$ is defined by sending an element x to the coefficient of t^i in the formal series

$$\gamma_t(x) = \sum_{j \geq 0} \lambda^j(x) \left(\frac{t}{1-t} \right)^j.$$

The γ-filtration on $K(X)$ is defined as $\gamma^0 = K(X)$, $\gamma^1 = \ker(\mathrm{rk})$ where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the map sending the class of a locally free sheaf $\mathcal{F}$ to its rank, and γ^i for $i \geq 2$ is generated by monomials $\gamma^{i_1}(x_1) \cdots \gamma^{i_r}(x_r)$ for any $r \geq 0$, $i_1 + \cdots + i_r \geq i$ and $x_1, ..., x_r$ elements of γ^1. We use the notation

$$\mathrm{gr}^i_\gamma K(X) := \gamma^{i/i+1} := \gamma^i/\gamma^{i+1} \quad \text{and} \quad \mathrm{gr}_\gamma K(X) := \bigoplus_{i \geq 0} \mathrm{gr}^i_\gamma K(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. When we need to be precise about which variety the γ-filtration is being considered for, we will specify by writing $\gamma^i(X)$ to mean the ith piece of the γ-filtration for the variety X. For further properties of these operations we refer to the references [Man69, MR071].

For the second, recall the coniveau filtration on $G(X)$ is defined by setting τ^i, for any $i \geq 0$, to be the ideal

$$\tau^i := \sum_{x \in X^{(j)}} \ker\left(G(X) \to G(X \setminus \overline{x})\right)$$

where $j \geq i$, $X^{(j)}$ denotes the set of codimension j points of X, and the arrows are flat pullbacks with respect to the respective inclusions $X \setminus \overline{x} \subset X$ for varying points x. We use the notation

$$\mathrm{gr}^i_\tau G(X) := \tau^{i/i+1} := \tau^i/\tau^{i+1} \quad \text{and} \quad \mathrm{gr}_\tau G(X) := \bigoplus_{i \geq 0} \mathrm{gr}^i_\tau G(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. Sometimes when more precision is needed, we include the variety in our notation for the coniveau filtration, i.e. $\tau^i(X)$ for the ith piece of the coniveau filtration of X.

The two filtrations are related:

Theorem 3.1. *Identify $K(X)$ with $G(X)$ under the canonical isomorphism. Then, for any $i \geq 0$ we have $\gamma^i \subset \tau^i$. Hence the isomorphism $K(X) \to G(X)$ induces a (graded) filtration-comparison morphism $\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\tau G(X)$. Moreover, if the filtration-comparison map is surjective, or injective, then the two filtrations are equal, i.e. $\gamma^i = \tau^i$ for all $i \geq 0$ (in particular, if either of these conditions hold then the filtration-comparison map is bijective).*

Proof. For the first claim, see [Man69]. The second claim about surjectivity implying bijectivity originally appears in [Kar98] and is updated in [KM18, Proposition 3.3] where the claim about injectivity implying bijectivity also appears. □

4. REDUCTIONS

The main purpose of this section is to provide a way to reduce computations of the associated graded ring for the γ-filtration of a Severi-Brauer variety X to the case $X = \mathrm{SB}(A)$ for a p-primary division algebra A. In this regard we utilize heavily the motivic techniques of Karpenko (e.g. [Kar95a, Corollary 1.3.2],[Kar17a, Lemma 3.5]). The reason we can use these results is due to the observation that for any Severi-Brauer variety X there is a Severi-Brauer variety Y so that the γ-filtrations of X and Y are the same and, simultaneously, the γ-filtration and coniveau filtration for this Y are isomorphic as well. This allows us to prove results about X by first replacing it with a functorially-nicer Y and then reducing to previously known results. This observation seems nice enough to name it.

Definition 4.1. Let X be an arbitrary smooth variety. We say that a smooth F-variety Y, with F being a field possibly different from k, is a *τ-functorial replacement of* X if the following conditions hold:

(1) there is an isomorphism of groups
$$\mathrm{coker}\left(K(X) \to K(\overline{X})\right) = \mathrm{coker}\left(K(Y) \to K(\overline{Y})\right)$$
where the arrows are pullbacks along the projections,
(2) there is an isomorphism of graded rings $\mathrm{gr}_\gamma K(X) = \mathrm{gr}_\gamma K(Y)$,
(3) the filtration-comparison map $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an isomorphism.

Remark 4.2. In the cases where we are concerned, condition (1) of Definition 4.1 will always imply condition (2) of Definition 4.1. Most likely, condition (2) also implies condition (1) in these cases.

Note also that it's important to allow the variation of the field of definition of Y. Often when these τ-functorial replacements are known to exist, the field F is a much larger field than k.

We're going to rephrase Definition 4.1 so that, when X is a Severi-Brauer variety, a τ-functorial replacement can be constructed using only data that one can read off from the associated central simple algebra. To do this we introduce the following definition which is a small generalization from one already in common use. From now on, we let A be an arbitrary central simple algebra and we set $X = \mathrm{SB}(A)$.

Definition 4.3. Suppose A has a decomposition $A = M_r(k) \otimes \left(\bigotimes_{p \text{ prime}} A_p\right)$ with each A_p a division algebra of p-primary power index. Then we define the *behavior* of A to be the sequence
$$\mathcal{B}eh(A) = \left(\mathrm{ind}(A), \mathrm{ind}(A^{\otimes 2}), \ldots, \mathrm{ind}(A^{\otimes \exp(A)})\right).$$
We define the *p-behavior*, where p is a specified prime, to be the sequence
$$\mathcal{B}eh(p, A) = \left(\mathrm{ind}(A_p), \mathrm{ind}(A_p^{\otimes p}), \ldots, \mathrm{ind}(A_p^{\otimes \exp(A_p)})\right).$$

Finally, we define the *reduced p-behavior* of A to be the sequence

$$r\mathcal{B}eh(p,A) = \left(v_p\mathrm{ind}(A_p), v_p\mathrm{ind}(A_p^{\otimes p}), \ldots, v_p\mathrm{ind}(A_p^{\otimes \exp(A_p)})\right).$$

If A is a p-primary algebra then, in order to relieve some notational burden, we will call the reduced p-behavior simply the reduced behavior of A, and we will write $r\mathcal{B}eh(A)$ instead of $r\mathcal{B}(p,A)$.

Remark 4.4. The reduced behavior is a strictly descending sequence ending in 0. Conversely, for every prime p and for every strictly descending sequence ending in 0 there is a p-primary algebra with reduced behavior the given sequence, see [Kar98, Lemma 3.10]. Note that it's possible to reconstruct the behavior of A from the p-behavior (or the reduced p-behavior) as p ranges over all primes.

An equivalent definition for a τ-functorial replacement Y of X, when Y is also a Severi-Brauer variety, is then:

Lemma 4.5. *A Severi-Brauer variety $Y = \mathrm{SB}(B)$ associated to a central simple algebra B is a τ-functorial replacement for X if, and only if, the following conditions hold:*

(1) $\deg(A) = \deg(B)$,
(2) *for every prime p, the reduced p-behaviors of A, B are the same $r\mathcal{B}eh(p,A) = r\mathcal{B}eh(p,B)$,*
(3) *the filtration-comparison map $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an isomorphism.*

Proof. For the forward direction, it suffices to observe that conditions (1) and (2) of the lemma imply condition (1) Definition 4.1 by Theorem 2.2. For condition (2) of Definition 4.1, this is observed in [IK99, Theorem 1.1 and Corollary 1.2].

For the reverse direction, we start by assuming $Y = \mathrm{SB}(B)$ is a τ-functorial replacement for X. Then

$$\deg(A) = \dim_{\mathbb{Q}}(\mathrm{gr}_\gamma K(X) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{gr}_\gamma K(Y) \otimes \mathbb{Q}) = \deg(B)$$

proves condition (1) of the lemma statement. To see that condition (2) of the lemma statement holds, one can use the fact that tensoring the cokernel with $\mathbb{Z}_{(p)}$ gives a decomposition

$$\mathrm{coker}\left(K(X) \to K(\overline{X})\right) \otimes \mathbb{Z}_{(p)} = (\mathbb{Z}/p^{n_0}\mathbb{Z})^{\oplus r_0} \oplus \cdots \oplus (\mathbb{Z}/p^{n_{m-1}}\mathbb{Z})^{\oplus r_{m-1}}$$

for integers $n_0 > \cdots > n_{m-1} > 0$. Then $r\mathcal{B}eh(p,A) = (n_0, ..., n_{m-1}, 0)$ and, as the same is true for Y and B, we find $r\mathcal{B}eh(p,A) = r\mathcal{B}eh(p,B)$ for every prime p. □

The remainder of this section is devoted to proving that, given an arbitrary Severi-Brauer variety like X, there exists a τ-functorial replacement Y of X such that Y is also a Severi-Brauer variety. Our starting point is that it's already known, from [Kar98, Theorem 3.7 and Lemma 3.10], that τ-functorial replacements exist for the Severi-Brauer varieties of p-primary division algebras for any prime p.

Lemma 4.6. *Fix a prime p and suppose that A is a division algebra with $\mathrm{ind}(A) = p^n$, for some $n \geq 0$. Then there exists a τ-functorial replacement Y for $X = \mathrm{SB}(A)$ such that Y is also a Severi-Brauer variety.*

Proof. This is a restatement of [Kar98, Theorem 3.7 and Lemma 3.10]. We recall for later use how one constructs such a replacement. Let B be a division algebra over a field F with

$$\mathrm{ind}(B) = \exp(B) = \mathrm{ind}(A).$$

Let $r\mathcal{B}eh(A) = (n_0, ..., n_m)$ be the reduced behavior of A. Set $Z_i = \mathrm{SB}(p^{n_i}, B^{\otimes p^i})$ to be the generalized Severi-Brauer variety of right ideals of $B^{\otimes p^i}$ of dimension $r_i = \deg(B^{\otimes p^i})p^{n_i}$ inside of $\mathrm{Gr}(r_i, B^{\otimes p^i})$. Let $Z = Z_1 \times \cdots \times Z_m$. Then the τ-functorial replacement constructed in [Kar98] is exactly $Y = \mathrm{SB}(B_{F(Z)})$. □

To extend this lemma to arbitrary Severi-Brauer varieties takes some effort. We first show that, if a Severi-Brauer variety X has a τ-functorial replacement Y that is also a Severi-Brauer variety, then every Severi-Brauer variety X' Brauer equivalent to X also has a τ-functorial replacement Y' that is a Severi-Brauer variety. Together with Lemma 4.6, this proves that τ-functorial replacements exist for the Severi-Brauer variety of any central simple algebra that has p-primary index for some prime p. To extend this result to Severi-Brauer varieties of arbitrary central simple algebras A with no conditions on the index, one replaces the primary division algebra factors of A (in a particular way) and then takes a matrix ring over the tensor product of these replacements. That the Severi-Brauer variety of this algebra is a τ-functorial replacement of the Severi-Brauer variety of our original algebra A is proved in Theorem 4.11 below.

We start with:

Lemma 4.7. *Suppose A is an arbitrary central simple algebra, with $X = \mathrm{SB}(A)$. Let D_A be the underlying division algebra of A and set $X' = \mathrm{SB}(D_A)$. Then the following statements hold.*

(1) *Suppose there exists a Severi-Brauer variety $Y = \mathrm{SB}(B)$ that is a τ-functorial replacement for X; if D_B is the underlying division algebra of B, then $Y' = \mathrm{SB}(D_B)$ is a τ-functorial replacement for X'.*

(2) *Suppose there exists a Severi-Brauer variety $Y' = \mathrm{SB}(D_B)$ that is a τ-functorial replacement for X'; if $B = M_r(D_B)$ for some r with $\deg(A) = \deg(B)$, then $Y = \mathrm{SB}(B)$ is a τ-functorial replacement for X.*

Proof. In both statements (1) and (2), it's clear conditions (1) and (2) of Lemma 4.5 hold for the algebra associated to the Severi-Brauer variety that we are trying to check is a τ-functorial replacement. So, we only need to check condition (3).

Note that the projections $Y \times Y' \to Y'$ and $Y \times Y' \to Y$ are both projective bundles over their targets. Thus the following diagram commutes where the vertical arrows are the filtration-comparison morphisms (or sums of these morphisms) and the horizontal equalities are from the projective bundle formulas for both $\mathrm{gr}_\gamma K$ and $\mathrm{gr}_\tau G$.

$$\begin{array}{ccccc}
\bigoplus \mathrm{gr}_\tau G(Y) & = & \mathrm{gr}_\tau G(Y \times Y') & = & \bigoplus \mathrm{gr}_\tau G(Y') \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus \mathrm{gr}_\gamma K(Y) & = & \mathrm{gr}_\gamma K(Y \times Y') & = & \bigoplus \mathrm{gr}_\gamma K(Y')
\end{array}$$

It follows if the left, or the right, vertical arrow is a surjection then the middle vertical arrow is a surjection and therefore, by Theorem 3.1, an isomorphism. If the middle vertical arrow of this diagram is an isomorphism, then the outer two vertical arrows are isomorphisms as well. Hence the left vertical arrow is an isomorphism if and only if the right vertical arrow is an isomorphism, as claimed. □

As a consequence of the above proof we get:

Proposition 4.8. *Suppose A has p-primary index. Then there exists a Severi-Brauer variety Y that is a τ-functorial replacement for $X = \mathrm{SB}(A)$.* □

To extend Proposition 4.8 to an arbitrary central simple algebra (with no requirements on the index) we'll need the following description of the p-torsion in $\mathrm{gr}_\gamma K(X)$.

Lemma 4.9. *Fix a prime p. Let F be a finite field extension of k with degree $[F : k]$ not divisible by p. Then the pullback along the projection $K(X) \to K(X_F)$ induces an isomorphism*

$$\mathrm{gr}_\gamma K(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{gr}_\gamma K(X_F) \otimes \mathbb{Z}_{(p)}.$$

Proof. Note that, by the projection formula, the pullback composed with the pushforward

$$K(X)\otimes\mathbb{Z}_{(p)}\to K(X_F)\otimes\mathbb{Z}_{(p)}\to K(X)\otimes\mathbb{Z}_{(p)}$$

is an isomorphism. As these are morphisms between free $\mathbb{Z}_{(p)}$-modules of the same rank, the composition in the other direction

$$K(X_F)\otimes\mathbb{Z}_{(p)}\to K(X)\otimes\mathbb{Z}_{(p)}\to K(X_F)\otimes\mathbb{Z}_{(p)}$$

is also an isomorphism. Thus the pullback itself $K(X)\otimes\mathbb{Z}_{(p)}\to K(X_F)\otimes\mathbb{Z}_{(p)}$ is an isomorphism.

Consider the following commuting ladder with exact rows and vertical arrows induced by the pullback along the projection $X_F\to X$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \gamma^{i+1}(X)\otimes\mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i}(X)\otimes\mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i/i+1}(X)\otimes\mathbb{Z}_{(p)} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle\pi_{i+1}} & & \downarrow{\scriptstyle\pi_{i}} & & \downarrow{\scriptstyle\pi_{i/i+1}} & & \\ 0 & \longrightarrow & \gamma^{i+1}(X_F)\otimes\mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i}(X_F)\otimes\mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i/i+1}(X_F)\otimes\mathbb{Z}_{(p)} & \longrightarrow & 0 \end{array}$$

The right vertical arrow is a surjection since $K(X)\otimes\mathbb{Z}_{(p)}\to K(X_F)\otimes\mathbb{Z}_{(p)}$ is a surjection (cf. [Mac18, Proof of Lemma 2.3]). By the snake lemma one gets short exact sequences

$$0\to\ker(\pi_{i/i+1})\to\operatorname{coker}(\pi_{i+1})\to\operatorname{coker}(\pi_i)\to 0.$$

Since for $j\geq\dim(X)$ one has $\operatorname{coker}(\pi_j)=0$, the claim follows. □

Remark 4.10. The above proof can be adapted to show that, for any morphism $X\to Y$ between smooth varieties X and Y, if the pullback $K(Y)\to K(X)$ is an isomorphism (resp. an isomorphism after tensoring with a flat ring R) then the pullback $\mathrm{gr}_\gamma K(Y)\to\mathrm{gr}_\gamma K(X)$ is an isomorphism (resp. an isomorphism after tensoring with a flat ring R); cf. [KM18, Lemma 3.6].

Finally, we've arrived at the main result of this section.

Theorem 4.11. *For an arbitrary central simple algebra A, there exists a Severi-Brauer variety Y that is a τ-functorial replacement for $X=\mathrm{SB}(A)$.*

Proof. Let $A=M_r(k)\otimes\left(\bigotimes_{p\text{ prime}}A_p\right)$ be a decomposition with each A_p a p-primary division algebra. We set $X_p=\mathrm{SB}(A_p)$ in the following.

Find a field F (e.g. $F=\mathbb{Q}$ works) such that, for each prime p, there exists division algebras B_p over F with $\mathrm{ind}(B_p)=\exp(B_p)=\mathrm{ind}(A_p)$. Fix any particular choice of prime p and consider the reduced p-behavior of A,

$$r\mathcal{B}eh(p,A)=(n_0,...,n_m).$$

Set $Z_p^i=\mathrm{SB}(p^{n_i},B_p^{\otimes p^i})$ as in Lemma 4.6. We set $Z_p=Z_p^1\times\cdots\times Z_p^m$ to be the product of these varieties. In a similar fashion we construct varieties Z_q for all other primes $q\neq p$. Let

$$Z=\prod_{p\text{ prime}}Z_p\quad\text{and}\quad Z^p=\prod_{q\text{ prime},\,q\neq p}Z_q$$

be the given products. Finally, set $B=M_r(F)\otimes\left(\bigotimes_{p\text{ prime}}B_p\right)$.

We claim $Y=\mathrm{SB}(B_{F(Z)})$ is a τ-functorial replacement for X. The proof proceeds in several steps. The first step we take is to show that

$$\text{(ts)}\qquad (B_p\otimes_F F(Z^p))\otimes_{F(Z^p)}F(Z_{p,F(Z^p)})=B_p\otimes_F F(Z)$$

is a τ-functorial replacement for A_p. But this is clear since, by index reduction [MPW96, equation (0.3)] one has

$$\mathrm{ind}(B_{p,F(Z^p)})=\exp(B_{p,F(Z^p)})=\mathrm{ind}(A_p)$$

and the left side of the equation (ts) is the algebra constructed exactly as in Lemma 4.6.

The next step we take is to show that condition (3) of Lemma 4.5 is satisfied by $Y = \mathrm{SB}(B_{F(Z)})$. Since conditions (1) and (2) are clear for $B_{F(Z)}$ (applying again index reduction in the same way as in Lemma 4.5), this will complete the proof of the theorem. To do this, we let L_p be a finite field extension of $F(Z)$ that splits B_q for all $q \neq p$ and such that $[L_p : F(Z)]$ is not divisible by p. Then the following square is commuting

$$\text{(D)} \qquad \begin{array}{ccc} \mathrm{gr}_\gamma K(Y) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{gr}_\gamma K(Y_{L_p}) \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathrm{gr}_\tau G(Y) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{gr}_\tau G(Y_{L_p}) \otimes \mathbb{Z}_{(p)} \end{array}$$

where the vertical arrows are filtration-comparison maps and the horizontal arrows are pullbacks with respect to the projection $Y_{L_p} \to Y$. Since the top horizontal arrow of (D) is an isomorphism by Lemma 4.9 and the right vertical arrow is an isomorphism, by the construction of Y_{L_p} and by the proof of Proposition 4.8, it follows that the left vertical arrow of (D) is an injection. Repeating this argument for all primes p allows us to conclude that the morphism $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an injection since it is after localizing at every maximal ideal of $\mathbb{Z}$. But, the filtration-comparison map has the nice property that injectivity implies surjectivity, see Theorem 3.1, which completes the proof. □

As an application of the existence of τ-functorial replacements for an arbitrary Severi-Brauer variety, let us show one way to extend known motivic results on Severi-Brauer varieties to statements for the associated graded ring of the γ-filtration.

Corollary 4.12. *Suppose A is an arbitrary central simple algebra and let D_A be the underlying division algebra of A. Write $X = \mathrm{SB}(A)$ and $X' = \mathrm{SB}(D_A)$. Then there is an isomorphism*

$$\bigoplus_{i=1}^{\deg(A)/\deg(D_A)} \mathrm{gr}_\gamma K(X') \to \mathrm{gr}_\gamma K(X).$$

Proof. Let $Y = \mathrm{SB}(B)$ be a τ-functorial replacement for X. Let $Y' = \mathrm{SB}(D_B)$ where D_B is the underlying division algebra for B. Let $r = \deg(A)/\deg(D_A) = \deg(B)/\deg(D_B)$. Then there is a canonical chain of isomorphisms,

$$\bigoplus_{i=1}^{r} \mathrm{gr}_\gamma K(X') \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathrm{gr}_\gamma K(Y') \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathrm{gr}_\tau G(Y') \xrightarrow{\sim} G(Y) \xleftarrow{\sim} \mathrm{gr}_\gamma K(Y) \xleftarrow{\sim} \mathrm{gr}_\gamma K(X)$$

using the definition of τ-functorial replacements and [Kar95a, Corollary 1.3.2], that defines the isomorphism of the Corollary. □

We end this section by making the following observation that generalizes the existence of a τ-functorial replacement to some other generalized flag varieties.

Corollary 4.13. *Let A be a central simple algebra with $\mathrm{ind}(A) = n$. Let $V_{i_1,\dots,i_r}(A)$ be the variety of flags of ideals in A of reduced dimensions $i_1, \dots, i_r$. If $\gcd(i_1, \dots, i_r, n) = 1$ then there exists a τ-functorial replacement for $V_{i_1,\dots,i_r}(A)$. Moreover, these τ-functorial replacements can be constructed as twisted flag varieties of the same kind.*

Proof. Let $Y = V_{i_1,\dots,i_r}(A)$ and $X = \mathrm{SB}(A)$. Let X' be a Severi-Brauer variety that is a τ-functorial replacement of X, using Theorem 4.11, and let B be the central simple algebra corresponding to X'. Let $Y' = V_{i_1,\dots,i_r}(B)$. Note that one has $\mathcal{B}eh(A) = \mathcal{B}eh(B)$. So, by the results of [Pan94], one also has

$$\mathrm{coker}\left(K(Y) \to K(\overline{Y})\right) = \mathrm{coker}\left(K(Y') \to K(\overline{Y'})\right)$$

and $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\gamma K(Y')$ for exactly the same reasons as when X is a Severi-Brauer. It remains to show the γ-filtration and coniveau filtration for Y' are equal.

To finish the proof, we're going to show $\mathrm{gr}_\tau G(Y')$ is generated by Chern classes. It follows from this that the canonical map $\mathrm{gr}_\gamma K(Y') \to \mathrm{gr}_\tau G(Y')$ is a surjection and therefore also an injection by Theorem 3.1. By [PSZ08, Corollary 3.4] the projection $X' \times Y' \to X'$ is a cellular fiber bundle over X'. It follows that $\bigoplus \mathrm{gr}_\tau G(X') = \mathrm{gr}_\tau G(X' \times Y')$ is generated by Chern classes. Again by [PSZ08, Corollary 3.4] the projection $X' \times Y' \to Y'$ is a projective bundle, and it follows that $\mathrm{gr}_\tau G(Y') \subset \bigoplus \mathrm{gr}_\tau G(Y') = \mathrm{gr}_\tau G(X' \times Y')$ is also generated by Chern classes. □

5. Describing the γ-filtration

The goal of this section is to prove our main result, Theorem 5.1, that computes some of the graded groups associated to the γ-filtration in low homological degree for a Severi-Brauer variety $X = \mathrm{SB}(A)$ associated to a central simple algebra A with p-primary index.

Theorem 5.1. *Let A be a central simple algebra with $\mathrm{ind}(A) = p^n$ and set $X = \mathrm{SB}(A)$. Then there are equalities*

$$\mathrm{gr}_\gamma^{p^n-i} K(X) = p^n(\xi - 1)^{p^n-i}\mathbb{Z}$$

for all $1 \leq i \leq p-1$.

In the above we're identifying $K(X)$ with its image in $K(X_F)$ for some splitting field F of A and we are setting ξ to be the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$ as in Theorem 2.2.

Using results of the previous section, this computation immediately generalizes to an arbitrary central simple algebra and to more general twisted flag varieties.

Our proof of the main theorem works in the following way. We first consider the filtration on $K(X)$ generated by K-theoretic Chern classes in $\zeta_X(1)$ and show that, inside of $K(X_F)$, this filtration is especially simple. Specifically, in low degrees this filtration is spanned by polynomials $p^n(\xi - 1)^i$ for large i. Then we write out a general generator of the γ-filtration on X in the same degree, considered also inside of $K(X_F)$, and show that p^n divides the coefficient of this general element. It follows that the γ-filtration is actually spanned, in these degrees, by K-theoretic Chern classes in $\zeta_X(1)$ and this allows us to conclude. The proof itself is long but entirely elementary.

Before proving this theorem, however, we describe a particular generating set for the γ-filtration for a Severi-Brauer variety $X = \mathrm{SB}(A)$ when A is a central simple algebra with p-primary index. This generating set appears in the literature already [Kar98, Bae15] but, the justification for why it exists is conceptually clearer using the arguments given here. We're also going to take this chance to uniformize the notation that will be used throughout the remainder of this text. Recall then the following definition [KM19, Definition A.1].

Definition 5.2. Let A be a central simple algebra with $\mathrm{ind}(A) = p^n$ and let $X = \mathrm{SB}(A)$. Let

$$S_X = \{i : v_p\mathrm{ind}(A^{\otimes p^i}) < v_p\mathrm{ind}(A^{\otimes p^{i-1}}) - 1\}$$

be the given set of natural numbers. We call the cardinality $\#S_X$ the *level of A* or the *level of X*.

In other words, the level of A is the number of places where the reduced behavior decreases by more than one from one position to the next. The relevance of the level is contained in the following lemma.

Lemma 5.3. [KM19, Lemma A.6] *The ring $K(X)$ is generated, as a λ-ring, by the classes of the sheaves of the set $\{\zeta_X(p^i)\}_i$ where i is an index for the set $\{0\} \cup S_X$.*

In particular, the above lemma implies the following lemma about a small generating set for the γ-filtration on $K(X)$.

Lemma 5.4. *Let A be a central simple algebra with p-primary index for a prime p. Set $X = \mathrm{SB}(A)$ to be the associated Severi-Brauer variety. Then the ith piece of the γ-filtration, $\gamma^i \subset K(X)$, is generated additively by products*

$$\gamma^{j_1}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \mathrm{rk}(x_r))$$

where $j_1 + \cdots + j_r \geq i$ and $x_1, ..., x_r$ are elements of $\{\zeta_X(p^i)\}_i$ where i indexes the set $S_X \cup \{0\}$.

Proof. Note that the images of these products generate the graded group γ^i/γ^{i+1} since these are the images of K-theoretic Chern classes of the negatives of the duals of the sheaves $\zeta_X(p^i)$. The claim can then be obtained by descending induction since for $i = \dim(X)$ one has $\gamma^{i-1/i} = \gamma^{i-1}$. □

Using the description of Theorem 2.2, the products appearing in the statement of Lemma 5.4 can be computed like so.

Lemma 5.5. *Let A be a central simple algebra with p-primary index for some prime p. Assume A has reduced behavior $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Fix a splitting field F of A and identify $K(X)$ with its image in $K(X_F)$. Let ξ be the class of $\mathcal{O}_{X_F}(-1)$.*

Then

$$\gamma^i(\zeta_X(p^j) - p^{n_j}) = \binom{p^{n_j}}{i}(\xi^{p^j} - 1)^i.$$

Proof. This is computed in [Kar98]. It's done by observing

$$\gamma_t(p^{n_j}\xi^{p^j} - p^{n_j}) = \gamma_t(p^{n_j}(\xi^{p^j} - 1)) = \gamma_t(\xi^{p^j} - 1)^{p^{n_j}} = (1 + (\xi^{p^j} - 1)t)^{p^{n_j}}$$

which gives the claim. □

We're almost in position to prove Theorem 5.1. The last ingredient we need for the proof is contained in the next definition and the following lemma.

Definition 5.6. Let $X = \mathrm{SB}(A)$ be the Severi-Brauer variety of a central simple algebra A with $\mathrm{ind}(A) = p^n$ for some prime p. Let $\eta^i(X)$ be the ideal of $K(X)$ generated by monomials

$$\gamma^{j_1}(\zeta_X(1) - p^n) \cdots \gamma^{j_r}(\zeta_X(1) - p^n)$$

with $j_1 + \cdots + j_r \geq i$. When it's clear from context, we simply write η^i for $\eta^i(X)$.

Lemma 5.7. *Let $X = \mathrm{SB}(A)$ be the Severi-Brauer variety of a central division algebra A with $\mathrm{ind}(A) = p^n$. Let F be a splitting field for X and make the identifications of Theorem 2.2. Then η^i defines a descending ring filtration on $K(X)$ and, for every $i \geq 0$, one has*

$$\eta^i = \bigoplus_{j \geq i} p^{n - v_p(j)}(\xi - 1)^j\mathbb{Z}.$$

Proof. The claim about being a filtration is clear. For the equality, we do this by showing both sides include in the other. The reverse direction

$$\eta^i \supset \bigoplus_{j \geq i} p^{n - v_p(j)}(\xi - 1)^j\mathbb{Z}$$

is clear since, for all $0 \leq i \leq p^n$ one has

$$\gamma^1(\zeta_X(1) - p^n) = p^n(\xi - 1) \quad \text{and} \quad \gamma^i(\zeta_X(1) - p^n) = \binom{p^n}{i}(\xi - 1)^i,$$

by Lemma 5.5, and $\gcd(p^{in}, \binom{p^n}{i}) = p^{n - v_p(i)}$ by [Kar98, Lemma 3.5].

For the inclusion

$$\eta^i \subset \bigoplus_{j \geq i} p^{n - v_p(j)}(\xi - 1)^j\mathbb{Z}$$

we note that $p^{n-v_p(j)}$ divides $\binom{p^n}{j_1}\cdots\binom{p^n}{j_r}$ whenever $j_1+\cdots+j_r=j\le p^n$. Indeed, let $v=\min_s\{v_p(j_s)\}$ and suppose, without loss of generality, that $v=v_p(j_1)$. Then

$$\begin{aligned} v_p\left(\binom{p^n}{j_1}\cdots\binom{p^n}{j_r}\right) =& n-v+\sum_{s=2}^{r}(n-v_p(j_s)) \\ \geq& n-v_p(j) \end{aligned}$$

This inequality, together with the definition of η^i and Lemma 5.5, gives the result. □

Proof of Theorem 5.1. It suffices by Corollary 4.12 to assume A is a division algebra. Our proof works by showing p^n divides the coefficient of every element of $\gamma^{p^n-p+1}\supset\gamma^{p^n-i}$ when each of these elements is written as polynomial in $(1-\xi)$. Note since there are inclusions

$$\gamma^{p^n-p+1}(X)\subset\tau^{p^n-p+1}(X)\subset\tau^{p^n-p+1}(X_F)=(1-\xi)^{p^n-p+1}K(X_F),$$

we can write every element y of γ^{p^n-p+1} as a sum

$$y=\sum_{j=p^n-p+1}^{p^n-1}a_j(1-\xi)^j$$

for some integers a_j. After we show p^n divides each of these a_j, it follows that we have inclusions

$$\eta^{p^n-p+1}\subset\gamma^{p^n-p+1}\subset\eta^{p^n-p+1}$$

and this will end the proof.

Suppose then

$$y=\gamma^{j_1}(x_1-\mathrm{rk}(x_1))\cdots\gamma^{j_r}(x_r-\mathrm{rk}(x_r))$$

is an arbitrary monomial generating γ^{p^n-p+1} like those described in Lemma 5.4. We work in two cases: each of $x_1,...,x_k$ is equal to $\zeta_X(1)$ for some $1\le k\le r$ or $\zeta_X(1)$ does not appear among the $x_1,...,x_r$ at all.

Assuming we're in the former case, let us make one more reduction. We're trying to give a lower bound the p-adic valuation of the coefficient in an expansion of y. We're also assuming each of $x_1,...,x_k$ are equal to $\zeta_X(1)$ and, since

$$n-v_p(r)=v_p\left(\binom{p^n}{r}\right)\le v_p\left(\binom{p^n}{j_1}\cdots\binom{p^n}{j_k}\right)$$

when $r=j_1+\cdots+j_k$ (see the end of the proof of Lemma 5.7), we can therefore assume $k=1$.

With these assumptions we can expand y as follows

$$\begin{aligned} y=&\binom{p^n}{j_1}(\xi-1)^{j_1}\binom{p^{n-t_2}}{j_2}(\xi^{p^{s_2}}-1)^{j_2}\cdots\binom{p^{n-t_r}}{j_r}(\xi^{p^{s_r}}-1)^{j_r} \\ =&\binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2}\cdots\binom{p^{n-t_r}}{j_r}(\xi-1)^{j_1}(\xi^{p^{s_2}}-1)^{j_2}\cdots(\xi^{p^{s_r}}-1)^{j_r} \end{aligned}$$

where here we are writing $x_k=\zeta_X(p^{s_k})$ for some integers $s_2,...,s_r\ge 1$ and $p^{n-t_k}=\mathrm{ind}(A^{\otimes p^{s_k}})$. Now by Lemma 2.3, equation 2, there is an expansion, for each integer k satisfying $2\le k\le r$,

$$\xi^{p^{s_k}}-1=\sum_{i=1}^{p^{s_k}}(-1)^i\binom{p^{s_k}}{i}(1-x)^i.$$

We set $x_{low}(k)=\sum_{i=1}^{p-1}(-1)^i\binom{p^{s_k}}{i}(1-x)^i$ to be sum containing the small degree summands of this latter sum and $x_{high}(k)=\sum_{i=p}^{p^{s_k}}(-1)^i\binom{p^{s_k}}{i}(1-x)^i$ to be the sum containing the high degree summands. We still have an equality

$$\xi^{p^{s_k}} - 1 = x_{low}(k) + x_{high}(k)$$

for every $2 \leq k \leq r$ but it's useful to group the terms in this way since p divides each $x_{low}(k)$ but one doesn't necessarily have that p divides any $x_{high}(k)$.

Rewriting y in terms of the $x_{low}(k)$'s and $x_{high}(k)$'s gives

$$y = \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2}\cdots\binom{p^{n-t_r}}{j_r}(\xi-1)^{j_1}(x_{low}(2)+x_{high}(2))^{j_2}\cdots(x_{low}(r)+x_{high}(r))^{j_r}.$$

By applying the binomial theorem and expanding we get

$$\begin{aligned}\prod_{k=2}^{r}(x_{low}(k)+x_{high}(k))^{j_k} &= \prod_{k=2}^{r}\left(\sum_{l=0}^{j_k}\binom{j_k}{l}x_{low}(k)^l x_{high}(k)^{j_k-l}\right)\\ &= \left(x_{low}(2)^{j_k}\cdots x_{low}(r)^{j_r} + \sum_{k=2}^{r} x_{high}(k)q_k\right)\end{aligned}$$

where q_k is a polynomial in the terms $x_{low}(2), \ldots, x_{low}(r), x_{high}(2), \ldots, x_{high}(r)$. If $x_{high}(k) \neq 0$ then the lowest degree in $(1-\xi)$ of $x_{high}(k)$ is p, while the lowest degree of any $x_{low}(k)$ is 1. In particular, the lowest degree in $(1-\xi)$ of any $x_{high}(k)q_k$ is $j_2 + \cdots + (j_k - 1 + p) + \cdots + j_r$. After multiplying by $(\xi-1)^{j_1}$ it follows

$$(\xi-1)^{j_1}\prod_{k=2}^{r}(x_{low}(k)+x_{high}(k))^{j_k} = (\xi-1)^{j_1}x_{low}(2)^{j_k}\cdots x_{low}(r)^{j_r}$$

because of

$$j_1 + j_2 + \cdots + (j_k - 1 + p) + \cdots + j_r \geq p^n - p + 1 - 1 + p = p^n$$

and Theorem 2.2.

Thus we find

$$\begin{aligned}y &= \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2}\cdots\binom{p^{n-t_r}}{j_r}(\xi-1)^{j_1}x_{low}(2)^{j_2}\cdots x_{low}(r)^{j_r}\\ &= \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2}\cdots\binom{p^{n-t_r}}{j_r}p^{j_2+\cdots+j_r}(\xi-1)^{j_1}\left(\frac{x_{low}(2)}{p}\right)^{j_2}\cdots\left(\frac{x_{low}(r)}{p}\right)^{j_r}\end{aligned}$$

since each $x_{low}(k)$ is divisible by p.

Now set $\alpha = \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2}\cdots\binom{p^{n-t_r}}{j_r}p^{j_2+\cdots+j_r}$. We have

$$\begin{aligned}v_p(\alpha) =& n - v_p(j_1) + \sum_{k=2}^{r}(n - t_k - v_p(j_k)) + j_2 + \cdots + j_r\\ \geq& n - v_p(j_1) + j_2 + \cdots + j_r\end{aligned}$$

We finish by showing $n - v_p(j_1) + j_2 + \cdots + j_r \geq n$ for all possible $j_1, \ldots, j_r$ or, equivalently, assuming $j_1 + \cdots + j_r = p^n - i$ for some i with $0 < i < p$ we finish by showing

$$p^n - i \geq j_1 + v_p(j_1).$$

Assuming i is largest possible we can also show $p^n - p + 1 \geq j_1 + v_p(j_1)$. We can assume $v_p(j_1) > 0$ as otherwise p^n divides $\binom{p^n}{j_1}$. Hence we can assume $j_1 = a_1 p^{n-1} + \cdots + a_{n-r}p^r$ with $0 \leq a_1, \ldots, a_{n-r} < p$ and some minimal $r \geq 1$. This inequality becomes

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r}p^r + r.$$

We make one last approximation, and assume all $a_1, ..., a_{n-r}$ are equal $(p-1)$, as this is the largest they can be. We're left checking

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r} p^r + r = p^n - p^r + r.$$

Rearranging, we check

$$p^r - p \geq r - 1$$

which is clear if $r = 1$ and is the same as

$$\frac{p^r - p}{r-1} \geq 1$$

for $r > 1$. Using the mean value theorem, the left of this inequality equals $f'(c)$ for some c in the interval $[1, r]$ and $f(x) = p^x$. Since $f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1$ we've completed this case.

We still need to check the second case, when $\zeta_X(1)$ is not a part of the γ-operations of our monomial. Following the same process as before, we're left to check the inequality $p^n - i \geq n$ for $0 < i < p$. But this is also readily checked to be true: we can assume we want to show $p^n - p + 1 \geq n$; and $p^n - p \geq n - 1$ is the same (ignoring the $n = 1$ case which is trivial) as $\frac{p^n - p}{n-1} \geq 1$ which by the mean value theorem equals $f'(c)$ for some c in the interval $[1, n]$ and $f(x) = p^x$; for all such c we have $f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1$. □

We conclude with a series of corollaries that motivated this work.

Corollary 5.8. *Let B be a central simple algebra, and let $Y = \mathrm{SB}(B)$ be the Severi-Brauer variety of B. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. Then for every pair of primes $p, q \in \{p_1, ..., p_r\}$ with $p \geq q$, and for all integers i satisfying $1 \leq i \leq q - 1$*

$$\mathrm{gr}_\gamma^{d-i} K(Y) \otimes \mathbb{Z}_{(p)} = d(1-\xi)^{d-i}\mathbb{Z}_{(p)},$$

where ξ is the class of $\mathcal{O}_{X_F}(-1)$ when identifying $K(X) \subset K(X_F)$ for a splitting field F of X.

Proof. Apply Lemma 4.9, Corollary 4.12, and Theorem 5.1. □

Corollary 5.9. *Let B be a central simple algebra, and let $Y = \mathrm{SB}(B)$ be the Severi-Brauer variety of B. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. Then for all integers i satisfying $1 \leq i \leq p_1 - 1$*

$$\mathrm{gr}_\gamma^{d-i} K(Y) = d(1-\xi)^{d-i},$$

where ξ is the class of $\mathcal{O}_{X_F}(-1)$ when identifying $K(X) \subset K(X_F)$ for a splitting field F of X.

Proof. This is true after localizing at every prime p by Corollary 5.8, so it's true in general. □

Corollary 5.10. *Suppose B is an arbitrary central simple algebra and set $X = \mathrm{SB}(B)$. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$ If $\mathrm{CH}(X)$ is generated by Chern classes and if the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$, taking the class of an integral subvariety V to the class of its structure sheaf $[\mathcal{O}_V]$, is an isomorphism, then for every pair of primes $p, q \in \{p_1, ..., p_r\}$ with $p \geq q$,*

$$\mathrm{CH}_j(X) \otimes \mathbb{Z}_{(p)} = d\mathbb{Z}_{(p)} \quad \textit{for all } j \leq q - 2.$$

Proof. In this setting, the rings $\mathrm{CH}(X)$ and $\mathrm{gr}_\gamma K(X)$ are isomorphic, [Kar17c, Theorem 3.1]. □

Corollary 5.11. *Suppose B is an arbitrary central simple algebra and set $X = \mathrm{SB}(B)$. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$ If $\mathrm{CH}(X)$ is generated by Chern classes and if the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$ is an isomorphism, then*

$$\mathrm{CH}_j(X) = d\mathbb{Z} \quad \textit{for all } j \leq p_1 - 2.$$

Proof. This is true after localizing at every prime p by Corollary 5.10, so it's true in general. ☐

Remark 5.12. The conditions of Corollaries 5.10 and 5.11 hold, for example, when B is a central simple algebra corresponding to a generic Severi-Brauer variety, see [Kar17c]. These conditions also hold for a more general class of algebras, see [KM19]. In both of these cases, Corollaries 5.10 and 5.11 were already known to hold so, we've reproved and generalized this result.

These corollaries can also be extended to more general flag varieties by the following lemmas.

Lemma 5.13. *Let B be a central simple algebra with $\mathrm{ind}(B) = n$. Let $X = \mathrm{SB}(B)$ and let $Y = V_{i_1,..,i_r}(B)$ be the variety of flags of ideals in B of reduced dimensions $i_1, ..., i_r$. If $\gcd(i_1, ..., i_r, n) = 1$ then the following statements hold:*

1. *$\mathrm{CH}^i(X)$ is torsion free for all $i \leq j$ if, and only if, $\mathrm{CH}^i(Y)$ is torsion free for all $i \leq j$,*
2. *$\mathrm{CH}_i(X)$ is torsion free for all $i \leq j$ if, and only if, $\mathrm{CH}_i(Y)$ is torsion free for all $i \leq j$.*

Replacing everywhere CH *occurs with* $\mathrm{gr}_\tau G$ *the same statements hold.*

Proof. In this case one has that $X \times Y \to Y$ is a projective bundle over Y and $X \times Y \to X$ is a cellular fibration over X by [PSZ08, Corollary 3.4]. Therefore

$$\bigoplus \mathrm{CH}(X) = \mathrm{CH}(X \times Y) = \bigoplus \mathrm{CH}(Y)$$

and the claim follows by looking at torsion in the respective degrees. The same argument works replacing CH by $\mathrm{gr}_\tau G$. ☐

Lemma 5.14. *Let B be a central simple algebra with $\mathrm{ind}(B) = n$. Let $X = \mathrm{SB}(B)$ and let $Y = V_{i_1,..,i_r}(B)$ be the variety of flags of ideals in B of reduced dimensions $i_1, ..., i_r$. If $\gcd(i_1, ..., i_r, n) = 1$ then the following statements hold:*

1. *$\mathrm{gr}^i_\gamma K(X)$ is torsion free for all $i \leq j$ if, and only if, $\mathrm{gr}^i_\gamma K(Y)$ is torsion free for all $i \leq j$,*
2. *$\mathrm{gr}_{\gamma,i} K(X)$ is torsion free for all $i \leq j$ if, and only if, $\mathrm{gr}_{\gamma,i} K(Y)$ is torsion free for all $i \leq j$.*

Proof. First make a τ-functorial replacement of X with a Severi-Brauer variety X' associated to a central simple algebra C. Note that, by the proof of Corollary 4.13 the variety $Y' = V_{i_1,...,i_r}(C)$ is a τ-functorial replacement of Y. Since the associated graded ring for the γ-filtration doesn't change when making a τ-functorial replacement, the claim follows from Corollary 5.13 applied to $\mathrm{gr}_\gamma K(X') = \mathrm{gr}_\tau G(X')$ and $\mathrm{gr}_\gamma K(Y') = \mathrm{gr}_\tau G(Y')$. ☐

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