ALGEBRAIC CONNECTIVE K-THEORY OF A SEVERI–BRAUER VARIETY WITH PRESCRIBED REDUCED BEHAVIOR

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ABSTRACT. We show that Chow groups of low dimension cycles are torsion free for a class of sufficiently generic Severi–Brauer varieties. Using a recent result of Karpenko, this allows us to compute the algebraic connective K-theory in low degrees for the same class of varieties. Independently of these results, we show that the associated graded ring for the topological filtration on the Grothendieck ring is torsion free in the same degrees for an arbitrary Severi–Brauer variety.

1. INTRODUCTION

The goal of this paper is to determine some low degree algebraic connective K-groups for a class of generic Severi–Brauer varieties that have arbitrary reduced behavior (see Example 2.2). By a recent result of Karpenko [Kar20] this problem can be reduced to checking that the canonical surjection, from the Chow groups to the associated graded groups for the topological filtration on the Grothendieck group of coherent sheaves, is an isomorphism in these low degrees. This is accomplished by Theorem 3.1 which shows that the Chow groups of these generic Severi–Brauer varieties are torsion free.

Motivated by this result, we also make some observations on the structure of the algebraic connective K-theory of the Severi–Brauer varieties constructed in Section 2 and we give a presentation (due to Karpenko) of the topological filtration for an arbitrary Severi–Brauer variety.

Although the results of this paper are new, the techniques that go into their proofs have mostly appeared already in other places. This is especially true for our proof of Theorem 3.1 that employs a number

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of tools that have been developed in [Kar17b, KM19, Mac19]. In fact, in specific cases, Theorem 3.1 can already be found in the subtleties of these articles but, this was mostly overlooked when they were originally written.

The contents of this paper are structured as follows. In Section 2, we show how one can extend known constructions of Severi–Brauer varieties with prescribed reduced behavior to a construction where one also knows information about the generators of the Chow rings of these varieties.

Section 3 contains our proof of Theorem 3.1. Here we pull a number of results from other recent works of the author and Karpenko to get our computation of the Chow groups of a Severi–Brauer variety. We've tried to write this section so that it can be read independently of these other works but, for the full proof one will have to look elsewhere.

Finally, we conclude in Section 4 with some observations of independent interest on algebraic connective K-theory and on the topological filtration.

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Notation and Conventions.

- We fix an arbitrary base field k. When no confusion should arise, we may also use k as index.
- For any field F, an F-variety, or simply a variety if the field F is clear from context, is a separated scheme of finite type over F.
- For a central simple k-algebra A, we write $\mathbf{SB}(A)$ for the associated Severi–Brauer variety of minimal right ideals of A.
- For a Severi–Brauer variety X, we write $\mathbf{AZ}(X)$ for the associated central simple k-algebra corresponding to the endomorphism algebra $\mathrm{End}(\zeta_X^\vee)$ of the unique nontrivial extension
$$0 \to \Omega_X \to \zeta_X \to \mathcal{O}_X \to 0$$
of locally free sheaves on X.
- Lastly, if p is a prime then by v_p we mean the usual p-adic valuation.

2. Severi–Brauer varieties

Here we provide examples of Severi–Brauer variety with some nice properties. Recall from [Kar98, Definition 3.8] that the *reduced behavior* of a central simple k-algebra A with $\operatorname{ind}(A) = p^n$ for a prime p and $n \geq 1$ is the sequence

$$r\mathcal{B}eh(A) = \left(v_p \operatorname{ind}(A^{\otimes p^i})\right)_{i=0}^{v_p \exp(A)}.$$

Remark 2.1. Let p be a prime, let $n \geq 0$ be an integer, and let A be a central simple k-algebra of index $\operatorname{ind}(A) = p^n$. Then the index $\operatorname{ind}(A^{\otimes i})$ depends only on the p-adic valuation $v_p(i)$:

$$\operatorname{ind}(A^{\otimes i}) = \operatorname{ind}(A^{\otimes p^{v_p(i)}}).$$

For example, if i is prime-to-p, then $\operatorname{ind}(A^{\otimes i}) = p^n$. In this way one can reconstruct the index $\operatorname{ind}(A^{\otimes i})$, for any $i \geq 0$, from the reduced behavior.

Example 2.2. Given any sequence of integers $S = (n_i)_{i=0}^m$ with the property that $n = n_0 > n_1 > \cdots > n_m = 0$, one can construct a Severi–Brauer variety X^S associated to a division algebra A^S of index $\operatorname{ind}(A^S) = p^n$ with reduced behavior

$$r\mathcal{B}eh(X^S) = (n_0, n_1, ..., n_m).$$

In this example, we show that one can construct X^S defined over a large field extension K of the base field k so that X^S has the additional property that the Chow ring $\mathrm{CH}(X^S_{LK})$ of the scalar extension of X^S to the composite field LK is generated by CH-Chern classes for every algebraic extension L/k.

To start, we fix a power $r = p^n$ of a prime p, we set $G = \mathrm{PGL}_r$, and we choose a faithful representation $G \to \mathrm{GL}(V)$ of G into the general linear group of a finite dimensional k-vector space V. We write E for the generic fiber of the quotient map $\pi : \mathrm{GL}(V) \to \mathrm{GL}(V)/G$. The group G acts on $\mathbb{P}^r$ via left multiplication and we choose $P \subset G$ to be the stabilizer of a rational point under this action. The quotient E/P is a Severi–Brauer variety over the field $F = k(\mathrm{GL}(V)/G)$ and the central simple F-algebra $\mathbf{AZ}(E/P)$ associated to E/P is a division algebra having index and exponent equal p^n.

In particular, by [Kar98, Example 3.9] the reduced behavior of $\mathbf{AZ}(E/P)$ is

$$r\mathcal{B}eh(\mathbf{AZ}(E/P)) = (n, n-1, ..., 1, 0)$$

the sequence whose ith term, starting from $i = 0$, is $n - i$. Now for any strictly decreasing sequence like $S = (n_i)_{i=0}^m$ with $n_0 = n$ and $n_m = 0$ we can consider the varieties

$$Z = \prod_{i=0}^{m} Z_i \quad \text{and} \quad Z_i = \mathbf{SB}(p^{n_i}, \mathbf{AZ}(E/P)^{\otimes p^i})$$

with $\mathbf{SB}(p^{n_i}, \mathbf{AZ}(E/P)^{\otimes p^i})$ the generalized Severi-Brauer variety of reduced dimension-p^{n_i} right ideals inside of $\mathbf{AZ}(E/P)^{\otimes p^i}$.

We claim that the pair

$$X^S = (E/P)_{F(Z)} \quad \text{and} \quad A^S = \mathbf{AZ}(E/P)_{F(Z)}$$

defined over the field $K = F(Z)$ has the specified properties. First, by an application of index reduction, see [Kar98, Lemma 3.10], the K-division algebra A^S has reduced behavior

$$r\mathcal{B}eh(A^S) = (n_0, n_1, ..., n_m)$$

as claimed. Next, we observe:

(1) $\mathrm{CH}(E/P)$ is generated by CH-Chern classes by [Kar18, Proposition 6.1] and [Kar17a, Proof of Lemma 2.1],
(2) $\mathrm{CH}(E/P \times Z)$ is generated by CH-Chern classes as $E/P \times Z$ is a chain of Grassmannian bundles over E/P [Kar18, Proposition 6.1] and [Kar17a, Proof of Lemma 2.1],
(3) and $\mathrm{CH}(X^S)$ is generated by CH-Chern classes since X^S is the generic fiber of the projection $E/P \times Z \to Z$.

Finally, the proofs of (1), (2), and (3) above go through unchanged for the varieties E_{LF}/P_L, $E_{LF}/P_L \times Z_{LF}$, and X^S_{LK} for every algebraic extension L/k showing that X^S and A^S have all of the claimed properties.

3. Chow theory

Throughout this section we assume $X = \mathbf{SB}(A)$ is the Severi–Brauer variety associated to a central simple k-algebra A having index $\mathrm{ind}(A) = p^n$ for some prime p and for some $n \geq 1$. We also fix a chain of finite field extensions of k

$$k \subset L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset L_n$$

such that the degree $[L_0 : k]$ is prime-to-p and

$$[L_{i+1} : L_i] = p \quad \text{and} \quad \mathrm{ind}(A_{L_i}) = p^{n-i}.$$

We write $\mathrm{CH}_i(X)$ for the integral Chow group of dimension-i cycles on X modulo rational equivalence and $\mathrm{CH}^i(X)$ for the group of codimension-i cycles. The main result of this section is the following theorem.

Theorem 3.1. *Let $X = \mathbf{SB}(A)$ and assume $\mathrm{ind}(A) = p^n$ for some prime p and for some $n \geq 1$. Fix an integer $0 \leq i \leq p-2$ and assume $\mathrm{CH}_i(X)$ is generated by polynomials in* CH-*Chern classes. Then $\mathrm{CH}_i(X) = \mathbb{Z}$.*

Example 3.2. Let p be a prime and let S be a strictly decreasing sequence of nonnegative integers ending in 0. Let X^S be the Severi–Brauer variety constructed from p and S as in Example 2.2. Then $\mathrm{CH}(X^S)$ is generated as a ring by CH-Chern classes so that Theorem 3.1 applies for all $0 \leq i \leq p-2$.

Since the groups $\mathrm{CH}_i(X)$ can contain only p-primary torsion, it follows from a restriction-corestriction argument that $\mathrm{CH}_i(X)$ and $\mathrm{CH}_i(X_{L_0})$ coincide. To simplify our notation, we'll assume from now on that $k = L_0$ and we'll work with the localized groups $\mathrm{CH}_i(X) \otimes \mathbb{Z}_{(p)}$.

The proof of Theorem 3.1 in this setting requires a number of structure results for subrings $\mathrm{CT}(i;X)$ of $\mathrm{CH}(X)$ that are generated by particular CH-Chern classes. Our starting point will be these structure results.

Definition 3.3. For any $i \geq 0$, we let $\mathrm{CT}(i;X)$ be the subring of $\mathrm{CH}(X)$ with generators the Chern classes of $\zeta_X(i) = \zeta_X \otimes_{A^{\otimes i}} M_i$ where M_i is any simple left $A^{\otimes i}$-module. The ring $\mathrm{CT}(i;X)$ is canonically graded, and we write $\mathrm{CT}^j(i;X) \subset \mathrm{CH}^j(X)$ for its jth graded summand.

We recall that the groups $\mathrm{CT}^j(i;X)$ are torsion free and of rank one for any $0 \leq j \leq \dim(X)$. Further, each of the localizations $\mathrm{CT}^j(i;X) \otimes \mathbb{Z}_{(p)}$ has a canonical generator that we denote by $\tau_i(j)$:

Proposition 3.4 ([KM19, Proposition A.8])**.** *For any $i \geq 0$, the $\mathbb{Z}_{(p)}$-module $\mathrm{CT}(i;X) \otimes \mathbb{Z}_{(p)}$ is free. Moreover, for any j with $0 \leq j < \deg(A)$ the degree-j summand $\mathrm{CT}^j(i;X) \otimes \mathbb{Z}_{(p)}$ is generated by the element*

$$\tau_i(j) := c_{p^v}(\zeta_X(i))^{s_0} c_{s_1}(\zeta_X(i))$$

where p^v is the largest power of p dividing $\mathrm{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \leq s_1 < p^v$. □

When there's possible ambiguity for where these classes are defined (e.g. if we work over multiple different fields simultaneously) we'll include a superscript like $\tau_i^X(j)$ to mean these classes are defined inside $\mathrm{CT}(i;X) \otimes \mathbb{Z}_{(p)}$. We collect here a number of results on the rings $\mathrm{CT}(i;X)$.

Lemma 3.5 ([Mac19, Lemma 3.4])**.** *Let F/k be a finite field extension and $\pi_{F/k} : X_F \to X$ the projection. Then the composition*

$$\mathrm{CT}^j(i;X) \subset \mathrm{CH}^j(X) \xrightarrow{\pi_{F/k}^*} \mathrm{CH}^j(X_F)$$

of the inclusion and flat pullback $\pi_{F/k}^*$ has image contained in $\mathrm{CT}^j(i;X_F)$. Moreover, if the composition (resp. this composition with $\mathbb{Z}_{(p)}$-coefficients)

$$\mathrm{CT}^j(i;X_F) \subset \mathrm{CH}^j(X_F) \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X)$$

of the inclusion and proper pushforward $\pi_{F/k,*}$ has image contained in $\mathrm{CT}^j(i;X)$ (resp. has image contained in $\mathrm{CT}^j(i;X) \otimes \mathbb{Z}_{(p)}$) then the projection formula holds for $\pi_{F/k,*}, \pi_{F/k}^*$ and the compositions

$$\pi_{F/k}^* \circ \pi_{F/k,*} \quad \text{and} \quad \pi_{F/k,*} \circ \pi_{F/k}^*$$

are both multiplication by $[F : k]$. □

Lemma 3.6 ([Kar17b, Proposition 3.5] and [Mac19, Lemma 3.5])**.** *Let F/k be a finite extension splitting A and $\pi_{F/k} : X_F \to X$ the projection. Then the composition*

$$\mathrm{CT}^j(i;X_F) \subset \mathrm{CH}^j(X_F) \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X)$$

of the inclusion and the proper pushforward $\pi_{F/k,*}$ has image contained in $\mathrm{CT}^j(i;X)$ for any $j \geq 0$ and for any $i \geq 1$. □

Lemma 3.7 ([Kar17b, Proposition 3.5])**.** *Let $j \geq 0$ be an integer and, in the notation above, let $F = L_{n-v_p(j)}$. Then the composition*

$$\mathrm{CT}^j(1;X_F) = \mathrm{CH}^j(X_F) \xrightarrow{\pi_{L_F/k,*}} \mathrm{CH}^j(X)$$

has image contained in $\mathrm{CT}^j(1;X)$. Further, this composition is surjective onto $\mathrm{CT}^j(1;X)$. □

The proof of Theorem 3.1 follows a technique developed in [Kar17b] to compute the Chow ring of a generic Severi–Brauer variety. This technique was also used in [KM19] to compute the Chow ring of some Severi–Brauer varieties with prescribed reduced behavior. Essentially, we use the projection formula to show that any element of $\mathrm{CH}_i(X)\otimes\mathbb{Z}_{(p)}$ that can be realized as a polynomial of CH-Chern classes is a multiple of $\tau_1(\dim(X) - i)$ for any $0 \leq i \leq p-2$.

Lemma 3.8. *Let $S_X = \{i_1 < \cdots < i_k\}$ be the level set of X (*[Mac20, Def. 5.2]*). Let $j \geq 0$ be an integer and let α be an element of $\mathrm{CH}^j(X)\otimes \mathbb{Z}_{(p)}$ that can be realized as a polynomial in* CH*-Chern classes. Then α is contained in the subgroup of $\mathrm{CH}^j(X) \otimes \mathbb{Z}_{(p)}$ generated by elements*

$$\tau_1(a_0)\tau_{p^{i_1}}(a_1)\cdots\tau_{p^{i_k}}(a_k)$$

where $a_0 + \cdots + a_k = j$ *for some integers* $a_0, ..., a_k \geq 0$.

Proof. By [KM19, Proposition A.5], every polynomial in CH-Chern classes is a polynomial in the Chern classes of $\{\zeta_X(i)\}_i$ where $i \in S_X \cup \{0\}$. Grouping terms using Proposition 3.4 gives exactly these generators. $\square$

Now we separate the proof of Theorem 3.1 into two cases. We want to show that each generator from Lemma 3.8 is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. To do this we note that either $\tau_1(a_0)$ appears in such a generator with $a_0 > 0$ or this term doesn't appear at all. The latter case is easy to handle.

Lemma 3.9. *Let* $S = \{i_1 < \cdots < i_k\}$ *be an arbitrary collection of integers. Let* $a_1, ..., a_k \geq 0$ *be integers such that* $p^n - p + 1 \leq a_1 + \cdots + a_k$. *Then*

$$\tau_{p^{i_1}}(a_1) \cdots \tau_{p^{i_k}}(a_k)$$

is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$.

Proof. Since $a_1 + \cdots + a_k \geq p^n - p + 1$ by assumption, there must exist an index r with $a_r \geq p^{n-i_r}$. Indeed, if this was false then for every $1 \leq r \leq k$ there would be an inequality $a_r < p^{n-i_r}$. Since $p^{n-i_r} < p^{n-r}$ this would imply the inequality

$$a_1 + \cdots + a_k < \sum_{l=1}^{k} p^{n-l} < \sum_{l=1}^{n} p^{n-l} = \frac{p^n - 1}{p - 1}.$$

But since one also has

$$\frac{p^n - 1}{p - 1} < p^n - p + 1$$

for all $p > 2$ and for all $n > 1$ this can't happen.

Assume then that $a_r \geq p^{n-i_r}$ for some $1 \leq r \leq k$. It follows from the proof of [KM19, Corollary A.13] there is an element x in $\mathrm{CH}(X_{L_n})$ with

$$\pi_{L_n/k,*}(x) = \tau_{p^{i_r}}(a_r).$$

Applying the projection formula to the product $\tau_{p^{i_1}}(a_1) \cdots \tau_{p^{i_k}}(a_k)$ and using Lemma 3.6 gives the result. $\square$

On the other hand, if $\tau_1(a_0)$ appears in the product of such a generator then the same proof becomes a little more complicated. We'll need the following lemma.

Lemma 3.10. *Let $0 \leq j \leq n$ be an integer. Define*

$$n_j = v_p(\mathrm{rk}(\zeta_X(p^j))) \quad \textit{and} \quad a_{i,j} = v_p(\mathrm{rk}(\zeta_{X_{L_i}}(p^i))).$$

Let $\pi_{L_i/k} : X_{L_i} \to X$ be the projection along L_i/k. Then there's an equality

$$\pi^*_{L_i/k}(\tau^X_{p^j}(k)) = \beta^j_{i,k}\tau^{X_{L_i}}_{p^j}(k)$$

for some $\beta^j_{i,k} \in \mathbb{Z}_{(p)}$ with

$$v_p(\beta^j_{i,k}) = \begin{cases} n_j - a_{i,j} - v_p(k/p^{a_{i,j}}) & \textit{if } k = 0 \pmod{p^{a_{i,j}}} \\ n_j - a_{i,j} & \textit{if } k \neq 0 \pmod{p^{a_{i,j}}}. \end{cases}$$

Proof. The proof follows the lines of [KM19, Lemma A.12]. Pulling back the total CH-Chern polynomial of $\zeta_X(p^j)$ to L_i we find

$$\begin{aligned} \pi^*_{L_i/k}c_t(\zeta_X(p^j)) &= c_t(\zeta_{X_{L_i}}(p^j))^{p^{n_j - a_{i,j}}} \\ &= \left(1 + \tau^{X_{L_i}}_{p^j}(1)t + \cdots + \tau^{X_{L_i}}_{p^j}(p^{a_{i,j}})t^{p^{a_{i,j}}}\right)^{p^{n_j - a_{i,j}}} \\ &= 1 + \beta^j_{i,1}\tau^{X_{L_i}}_{p^j}(1)t + \cdots + \beta^j_{i,p^{n_j}}\tau^{X_{L_i}}_{p^j}(p^{n_j})t^{p^{n_j}}. \end{aligned}$$

The p-adic valuations of the $\beta^j_{i,k}$ can be determined by the multinomial formula. This is done explicitly in [KM19, Lemma B.4]: set $r = a_{i,j}$ and $s = n_j$. □

Proof of Theorem 3.1. We proceed by showing that every generator as in Lemma 3.8 is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. If in such a generator

$$\tau_1(a_0)\tau_{p^{i_1}}(a_1)\cdots\tau_{p^{i_k}}(a_k)$$

one has $a_0 = 0$ then we can apply Lemma 3.9. We're left in the case $a_0 > 0$.

Set $v = v_p(a_0)$ and let $F = L_{n-v}$ (if $v \geq n$ then the claim follows from the proof of [KM19, Corollary A.13]). Now Lemma 3.7 shows that there is an element $x \in \mathrm{CH}^{a_0}(X_F) \otimes \mathbb{Z}_{(p)}$ so that the pushforward of x along the projection $\pi_{F/k} : X_F \to X$ is

$$\pi_{F/k,*}(x) = \tau^X_1(a_0).$$

Since one has $\pi_{F/k,*} \circ \pi^*_{F/k} = [F : k] = p^{n-v}$ by construction of F and, one has

$$\pi^*_{F/k}(\tau^X_1(a_0)) = \beta^0_{n-v,a_0}\tau^{X_F}_1(a_0)$$

with $v_p(\beta^0_{n-v,a_0}) = n - v$ by Lemma 3.10, it follows that $x = \alpha\tau^{X_F}_1(a_0)$ for some $\alpha \in \mathbb{Z}_{(p)}$ not divisible by p.

Applying the projection formula one finds

$$\begin{aligned}\tau_1^X(a_0)\tau_{p^{i_1}}^X(a_1)\cdots\tau_{p^{i_k}}^X(a_k) &= \pi_{F/k,*}(\alpha\tau_1^{X_F}(a_0))\tau_{p^{i_1}}^X(a_1)\cdots\tau_{p^{i_k}}^X(a_k)\\ &= \pi_{F/k,*}\left(\alpha\tau_1^{X_F}(a_0)\cdot\pi_{F/k}^*\left(\tau_{p^{i_1}}^X(a_1)\cdots\tau_{p^{i_k}}^X(a_k)\right)\right)\\ &= \pi_{F/k,*}\left(\beta\alpha\tau_1^{X_F}(a_0)\tau_{p^{i_1}}^{X_F}(a_1)\cdots\tau_{p^{i_k}}^{X_F}(a_k)\right).\end{aligned}$$

We won't use it but, the coefficient β can also be explicitly determined by Lemma 3.10. We claim that

$$\tau_{p^{i_1}}^{X_F}(a_1)\cdots\tau_{p^{i_k}}^{X_F}(a_k)$$

is contained in $\pi_{L_n/F,*}\left(\mathrm{CH}(X_{L_n})\otimes\mathbb{Z}_{(p)}\right)\subset\mathrm{CT}(1;X_F)\otimes\mathbb{Z}_{(p)}$. Indeed, this is true by Lemma 3.9 since there is an integer $1\le r\le k$ with $a_r\ge p^{v-i_r}$. In particular, there is an element $y\in\mathrm{CH}(X_{L_n})\otimes\mathbb{Z}_{(p)}$ with

$$\begin{aligned}\tau_1^X(a_0)\tau_{p^{i_1}}^X(a_1)\cdots\tau_{p^{i_k}}^X(a_k) &= \pi_{F/k,*}\left(\beta\alpha\tau_1^{X_F}(a_0)\tau_{p^{i_1}}^{X_F}(a_1)\cdots\tau_{p^{i_k}}^{X_F}(a_k)\right)\\ &= \pi_{F/k,*}\left(\beta\alpha\tau_1^{X_F}(a_0)\pi_{L_n/F,*}(y)\right)\\ &= \pi_{F/k,*}\left(\pi_{L_n/F,*}\left(y\cdot\pi_{L_n/F}^*\left(\beta\alpha\tau_1^{X_F}(a_0)\right)\right)\right)\\ &= \pi_{L_n/k,*}\left(y\pi_{L_n/F}^*\left(\beta\alpha\tau_1^{X_F}(a_0)\right)\right)\end{aligned}$$

and this last element is contained in $\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)}$ by Lemma 3.6. □

4. Connective K-theory

Throughout this section we let X be an arbitrary k-variety. We write $\mathscr{M}(X)$ for the category of coherent sheaves on X. The category $\mathscr{M}(X)$ is abelian and admits a filtration

$$0=\mathscr{M}_{-1}(X)\subset\mathscr{M}_0(X)\subset\cdots\subset\mathscr{M}_n(X)=\mathscr{M}(X)$$

by Serre subcategories $\mathscr{M}_i(X)$ defined as the category of coherent sheaves on X supported in dimension-i or less. Each category $\mathscr{M}_i(X)$ is also abelian and it makes sense to consider the Grothendieck group $K(\mathscr{M}_i(X))$.

We refer to [Cai08] for our treatment of the algebraic connective K-theory $\mathrm{CK}(X)$ of X. By definition $\mathrm{CK}(X)$ is the sum of groups $\mathrm{CK}_i(X)$ that can be realized as the image of the induced map on K-groups under the exact inclusion $\mathscr{M}_i(X)\subset\mathscr{M}_{i+1}(X)$:

$$\mathrm{CK}(X)=\bigoplus_{i\in\mathbb{Z}}\mathrm{CK}_i(X)\quad\text{and}\quad\mathrm{CK}_i(X)=\mathrm{Im}\left(K(\mathscr{M}_i(X))\to K(\mathscr{M}_{i+1}(X))\right).$$

When X is smooth, connected, and of dimension-n the group $\mathrm{CK}(X)$ has the structure of a commutative and graded ring. In this case, we often write $\mathrm{CK}^i(X)$ for the group summand $\mathrm{CK}_{n-i}(X)$.

For any integer i, the group $\mathrm{CK}^i(X)$ has the structure of left $K(X)$-module (where $K(X)$ is the Grothendieck ring of locally free sheaves on X) induced by the tensor product of sheaves. Indeed, tensoring by a locally free sheaf is exact on $\mathscr{M}_{n-i}(X)$ and this descends to a morphism

$$K(X) \to \mathrm{End}(\mathrm{CK}^i(X))$$

sending a class $[\mathcal{F}]$ to the endomorphism sending $[\mathcal{G}]$ to $[\mathcal{F} \otimes \mathcal{G}]$.

In some instances, it's known that $\mathrm{CK}^i(X)$ can be naturally associated with a subgroup of the Grothendieck group $G(X)$ of coherent sheaves on X. More precisely, for any $i \in \mathbb{Z}$ one can consider the morphism

$$\psi_X^i : \mathrm{CK}^i(X) \to \tau^i(X)$$

to the ith piece of the topological filtration $\tau^i(X) \subset G(X)$ defined via the map

$$\mathrm{Im}\,(K(\mathscr{M}_{n-i}(X)) \to K(\mathscr{M}_{n-i+1}(X))) \to \mathrm{Im}\,(K(\mathscr{M}_{n-i}(X)) \to K(\mathscr{M}(X)))$$

induced by the inclusion $\mathscr{M}_{n-i+1}(X) \subset \mathscr{M}(X)$; for $i \leq 2$ the map ψ_X^i is an isomorphism [Kar20, Remark A.6]. In general, if we identify $K(X) = G(X)$ by the canonical map, then the $K(X)$-module structure on $\mathrm{CK}^i(X)$ is related to the ring structure of $\mathrm{CK}(X)$ via ψ_X^i.

Lemma 4.1. *Let β be the Bott element, i.e. the element of $\mathrm{CK}^{-1}(X)$ represented by the class of $\mathcal{O}_X$. Then the diagram below, with top horizontal arrow induced by the $K(X)$-module structure morphism on $\mathrm{CK}^i(X)$ and bottom horizontal arrow the ring structure map on $\mathrm{CK}(X)$, is commutative for every $i, j \in \mathbb{Z}$.*

$$\begin{tikzcd}
\tau^j(X) \otimes \mathrm{CK}^i(X) \arrow[r] & \mathrm{CK}^i(X) \\
\mathrm{CK}^j(X) \otimes \mathrm{CK}^i(X) \arrow[r] \arrow[u, "\psi_X^j \otimes 1"] & \mathrm{CK}^{i+j}(X) \arrow[u, "\cdot \beta^j"']
\end{tikzcd}$$

In particular, the composition

$$\tau^j(X) \otimes \mathrm{CK}^i(X) \to K(X) \otimes \mathrm{CK}^i(X) \to \mathrm{CK}^i(X)$$

has image contained in $\beta^j \mathrm{CK}^{i+j}(X)$.

Proof. This follows from the fact that the multiplication by β map and the map induced by the inclusion $\mathscr{M}_{n-i}(X) \to \mathscr{M}_{n-i+2}(X)$ are identical on algebraic connective K-theory, cf. [Cai08, Proof of Theorem 7.1]. In particular, one has

$$\psi^j(\mathcal{F}) \cdot \mathcal{G} = (\beta^j \cdot \mathcal{F}) \cdot \mathcal{G} = \beta^j \cdot (\mathcal{F} \cdot \mathcal{G}) = \psi^j(\mathcal{F} \cdot \mathcal{G})$$

for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ supported in codimension-j and codimension-i respectively. □

Alternatively, one can relate the $K(X)$-module structure on $\mathrm{CK}(X)$ to the ring structure in the following way. Give the tensor product $\mathrm{CK}^i(X) \otimes \mathrm{CK}^j(X)$ a $K(X)$-module structure for any $i, j \in \mathbb{Z}$ by acting on the left of either $\mathrm{CK}^i(X)$ or $\mathrm{CK}^j(X)$. The multiplication map is then a morphism of $K(X)$-modules.

The following lemma shows the relationship between K-Chern classes, CK-Chern classes, and the $K(X)$-module structure on $\mathrm{CK}(X)$.

Lemma 4.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two vector bundles on X. Then*

$$c_i^K(\mathcal{F}) \cdot c_j^{\mathrm{CK}}(\mathcal{G}) = \beta^i c_i^{\mathrm{CK}}(\mathcal{F}) c_j^{\mathrm{CK}}(\mathcal{G})$$

for any pair of integers $i, j \geq 0$.

Proof. The multiplication map

$$\mathrm{CK}^i(X) \otimes \mathrm{CK}^j(X) \to \mathrm{CK}^{i+j}(X)$$

is a morphism of left $K(X)$-modules when the tensor product is given its left $K(X)$-module structure. This means that, by possibly moving to a successive chain of projective bundles over X and applying the projective bundle formula [Cai08, Theorem 6.3], it suffices to check the formula when $i = j = 1$ and when $\mathcal{F}, \mathcal{G}$ are both line bundles; here the claim is obvious. □

Theorem 4.3. *Consider the following statements:*

(1) $\mathrm{CH}(X)$ *is generated as a ring by* $\mathrm{CH}^1(X)$
(2) $\mathrm{CK}(X)$ *is generated as a $K(X)$-algebra by* $\mathrm{CK}^1(X)$ *and* β
(3) $\mathrm{CK}^i(X)$ *is generated as a $K(X)$-module by polynomials of* CK-*Chern classes for every $i \in \mathbb{Z}$*
(4) $\mathrm{CK}^i(X)$ *is generated as a $K(X)$-module by polynomials of* CK-*Chern classes for some $i \in \mathbb{Z}$*
(5) $\mathrm{CK}^i(X)$ *is generated additively by polynomials of* CK-*Chern classes and* $\beta\mathrm{CK}^{i+1}(X)$ *for some $i \in \mathbb{Z}$*
(6) $\mathrm{CH}^i(X)$ *is generated additively by polynomials of* CH-*Chern classes for some $i \in \mathbb{Z}$.*

Then (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\iff$ (6).

Proof. (1) $\implies$ (2): For any $i \in \mathbb{Z}$ the exact sequence of [Cai08, Theorem 7.1]

$$0 \to \beta \mathrm{CK}^{i+1}(X) \to \mathrm{CK}^i(X) \to \mathrm{CH}^i(X) \to 0$$

shows that $\mathrm{CK}^i(X)$ is generated additively by polynomials of first CK-Chern classes (lifts of the generators of $\mathrm{CH}^i(X)$) and $\beta\mathrm{CK}^{i+1}(X)$. Now Lemma 4.2 shows that $\beta\mathrm{CK}^{i+1}(X)$ is generated as a $K(X)$-module by polynomials of first CK-Chern classes as well.

(2) $\implies$ (1): The canonical surjection $\mathrm{CK}(X) \to \mathrm{CH}(X)$ takes CK-Chern classes to CH-Chern classes and has kernel $\beta\mathrm{CK}(X)$. Normally, this would imply $\mathrm{CH}(X)$ is generated as a $K(X)$-algebra by $\mathrm{CH}^1(X)$ but, in this case any element of $K(X) = \mathbb{Z} \oplus \tau^1(X)$ acts on $\mathrm{CH}(X)$ only via by its rank because of Lemma 4.1.

(2) $\implies$ (3) $\implies$ (4): This is obvious.

(4) $\implies$ (5): Let $\{p_i\}_{i\in I}$ be a set of polynomials in CK-Chern classes that generate $\mathrm{CK}^i(X)$ as a $K(X)$-module. Then every element x of $K(X)$ can be written as $x = \mathrm{rk}(x) + (x - \mathrm{rk}(x))$ where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the rank map. The element $x - \mathrm{rk}(x)$ is contained in $\tau^1(X)$ so for any index i one finds

$$xp_i = \mathrm{rk}(x)p_i + (x - \mathrm{rk}(x))p_i = \mathrm{rk}(x)p_i + \beta z$$

for some element z of $\mathrm{CK}^{i+1}(X)$ by Lemma 4.1.

(5) $\iff$ (6): One can use again that the surjection $\mathrm{CK}^i(X) \to \mathrm{CH}^i(X)$ takes CK-Chern classes to CH-Chern classes and, by [Cai08, Theorem 7.1], has kernel $\beta\mathrm{CK}^{i+1}(X)$. □

Remark 4.4. In general, the implications (5) $\implies$ (4) $\implies$ (3) $\implies$ (2) between the properties of Theorem 4.3 are false.

Remark 4.5. Fix an integer ℓ. Suppose that X is a smooth and connected variety satisfying one of either (1)-(2), or (5)-(6) for all $i \geq \ell$, of Theorem 4.3. Then every projective bundle over X and every localization of X satisfies the same properties. To see this, one can use the projective bundle formula and the localization exact sequence for the Chow ring.

Theorem 4.3 can be combined with [Kar20, Theorem 2.2] and Theorem 3.1 to give the following corollary. We recall that there are canonical surjections

$$\psi_X^i : \mathrm{CK}^i(X) \to \tau^i(X) \quad \text{and} \quad \varphi_X^i : \mathrm{CH}^i(X) \to \mathrm{gr}_\tau^i G(X).$$

Here φ^i_X is defined by taking the class of an irreducible variety V to the class of the structure sheaf $[\mathcal{O}_V]$.

Corollary 4.6. *Let $X = \mathbf{SB}(A)$ be the Severi–Brauer variety associated to a central simple k-algebra A with $\operatorname{ind}(A) = p^n$ for some prime p and some integer $n \geq 0$. Assume that $\mathrm{CH}_i(X)$ is generated by polynomials of CH-Chern classes for all $0 \leq i \leq p-2$. Then the canonical morphism*

$$\psi_X^{\dim(X)-i} : \mathrm{CK}_i(X) \to \tau_i(X)$$

is an isomorphism for all $0 \leq i \leq p-2$, and $\mathrm{CK}_i(X)$ is generated by CK-Chern classes and $\beta\mathrm{CK}_{i-1}(X)$. □

Regardless of any information coming from φ^i_X we can still compute $\tau_i(X)$.

Theorem 4.7. *Let A be a division algebra with $\operatorname{ind}(A) = p^n$ for some prime p and some $n \geq 1$ and let $X = \mathbf{SB}(A)$. Identify $G(X_{\overline{k}}) \cong \mathbb{Z}[x]/(1-x)^{p^n}$ where $x = \mathcal{O}_{X_{\overline{k}}}(-1)$. Set $h = 1-x$ and identify $G(X) \subset G(X_{\overline{k}})$ via the pullback.*

Then the ith piece of the topological filtration $\tau_i(X)$ on $G(X)$ is

$$\tau_i(X) = \bigoplus_{j \leq i} p^n h^{p^n-1-j}\mathbb{Z} = p^n \tau_i(X_{\overline{k}})$$

for every $i \leq p-2$.

Proof. There's an equality

$$\tau_i(X_{\overline{k}}) = \bigoplus_{j \leq i} h^{p^n-1-j}\mathbb{Z}$$

so, in order to prove the result it suffices to show that $\tau_i(X) = p^n\tau_i(X_{\overline{k}})$. Note that the containment $p^n\tau_i(X_{\overline{k}}) \subset \tau_i(X)$ is proved in [Mac20, Lemma 5.7] so that we only need to show the reverse containment.

For any integer $0 \leq i \leq p-2$ (the nontrivial cases), an arbitrary element for $\tau_i(X) \subset \tau_i(X_{\overline{k}})$ is of the form

$$\sum_{j=p^n-1-i}^{p^n-1} s_j h^j$$

for some integers s_j. Writing this element as a sum of powers of elements $a_l(1-h)^{p^n-1-l}$ with $a_l = \operatorname{ind}(A^{\otimes l})$ (see [Kar98, Theorem 3.1]) gives an equality

$$\sum_{j=p^n-1-i}^{p^n-1} s_j h^j = \sum_{l=0}^{p^n-1} t_l a_l (1-h)^{p^n-1-l} \tag{no.1}$$

with t_l some integers. Since powers of h form a basis for $G(X_{\overline{k}})$ we can expand the right side of (no.1) and compare the coefficients of h^j to find

$$s_j = \sum_{l \geq j} t_l a_l \binom{p^n - 1 - l}{p^n - 1 - j}. \tag{no.2}$$

Now all of the a_l that appear in (no.2) have $p^n - 1 \geq l \geq j \geq p^n - p + 1$. Hence l is prime-to-p and $a_l = \mathrm{ind}(A^{\otimes l}) = p^n$ by Remark 2.1. It follows that $\tau_i(X)$ is contained in $p^n \tau_i(X_{\overline{k}})$ for every $0 \geq i \geq p-2$ as claimed. □

It follows immediately from Theorem 4.7 that one can describe the associated graded objects $\mathrm{gr}_{\tau,i} G(X)$ for the topological filtration on $G(X)$ in the degrees $0 \leq i \leq p-2$.

Corollary 4.8. *Let $X = \mathbf{SB}(A)$ be the Severi–Brauer variety associated to a central simple k-algebra A with index $\mathrm{ind}(A) = p^n$ for some prime p and some integer $n \geq 0$. Then $\mathrm{gr}_{\tau,i} G(X)$ is torsion free for all $0 \leq i \leq p-2$.* □

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