CODIMENSION 2 CYCLES ON SEVERI-BRAUER VARIETIES AND DECOMPOSABILITY

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Abstract. In this text we show that one can generalize results showing that $\mathrm{CH}^2(X)$, for various Severi-Brauer varieties X, is sometimes torsion free. In particular we show that for any pair of odd integers (n, m), with m dividing n and sharing the same prime factors, one can find a central simple k-algebra A of index n and exponent m that moreover has $\mathrm{CH}^2(X)$ torsion free for $X = \mathrm{SB}(A)$. One can even take $k = \mathbb{Q}$ in this construction.

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise.

If p is a prime we write v_p for the p-adic valuation.

1. Introduction

Severi-Brauer varieties are a class of objects that are incredibly interesting, from the point of view of intersection theory and, particularly, to the author of the present text. A Severi-Brauer variety X is a twisted form of projective space, meaning that it becomes isomorphic to projective space after moving over some finite extension of the base field. And while the intersection theory of projective space is the simplest of examples, a Severi-Brauer variety may be one of the most complicated, but still one of the most accessible, varieties where a complete computation of the Chow ring seems possible.

The reason Severi-Brauer varieties seem accessible, from an intersection theory point of view, is because they are related (equivalent) to the class of central simple algebras. More precisely, one can associate to any central simple algebra A a Severi-Brauer variety $X = \mathrm{SB}(A)$ as the variety of minimal left ideals inside of A (as a subvariety of a Grassmannian). One can then expect to get relations in the geometry of X by knowing certain algebraic properties about A. The topic of this text is aimed towards extending techniques showing that the Chow group of codimension 2 cycles on X is often simpler when A is decomposable into a tensor product of smaller algebras.

The reason Severi-Brauer varieties are still difficult objects to study is largely because the nuanced structure of central simple algebras is still mysterious. For example, one can associate to any central simple algebra A, whose dimension is a power of a prime p, a sequence of integers depending on the dimension of the division algebra Brauer equivalent to the ith tensor power $A^{\otimes i}$ for varying $i \geq 0$. (The dimension of the division algebra equivalent to A is called the index of A, denoted $\mathrm{ind}(A)$, and one only needs to know these indices up to the exponent $\exp(A)$ of A, i.e. the smallest $i > 0$ such that $A^{\otimes i}$ is isomorphic with a matrix ring). This sequence, called the reduced behavior of A, already completely determines the structure of the Grothendieck ring of locally free sheaves on X (see Section 2 below for more details on this relation, background, and more).

One could hope that this sequence would also completely determine the structure of the Chow ring of X. But, it was shown in [Kar98] that this was not the case: there are two algebras A, B (over different fields) with associated Severi-Brauer varieties $X = \mathrm{SB}(A)$ and $Y = \mathrm{SB}(B)$, whose

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reduced behaviors are the same but, in the group $\mathrm{CH}^2(X)$ there is nontrivial torsion and, on the other hand, the group $\mathrm{CH}^2(Y)$ is torsion free.

The precise observations that differentiated the Chow groups of the varieties X, Y above were the following. First, X was generic in some sense. This meant that the graded ring associated to the gamma filtration on the Grothendieck ring of locally free sheaves on X agreed with the graded ring associated to the topological, or coniveau, filtration on the Grothendieck ring of coherent sheaves on X. Karpenko then computed the associated graded for the gamma filtration explicitly and checked that it contained nontrivial torsion. By contrast, Y was associated to an algebra B that was decomposable – the opposite of what one would expect to be a generic property (if one thinks of a generic property as one that somehow specializes to every other example).

The examples of [Kar98] were an accumulation of a lot of work that had been done on the two associated graded rings for the Grothendieck ring of a Severi-Brauer variety (see [Kar95a, Kar96, Kar95b]). The proofs of [Kar98] are more or less elementary but, there were some unfortunate features to these examples: the computations one needed to do could be unpleasant in any sort of generality and, they often relied on the assumption that the Chow ring was isomorphic to a simpler ring which, by [Mer95], didn't always happen.

In the present text we (try to) suggest a way that one can avoid the use of the associated graded rings of the Grothendieck ring. Every computation in the present text, with the exception of Proposition 3.6, focuses on working solely in the Chow ring. And, although our examples rely on results from [Kar98], these too can be recovered from the methods used here.

The main result of this text is the statement of the abstract. Essentially this text shows that for any pair of odd integers (n, m), and assuming m divides n and that n, m have the same prime factors, there is an algebra A having index n and exponent m such that $\mathrm{CH}^2(X)$ is torsion free where $X = \mathrm{SB}(A)$. This compliments the fact that it's known, for example from [Kar98] together with [Kar17], that for any pair of odd integers (n, m), with m dividing n and having the same prime factors, there exists an algebra B having index n and exponent m such that, for $Y = \mathrm{SB}(B)$, the group $\mathrm{CH}^2(Y)$ contains nontrivial torsion.

The techniques used in this text are, to the authors' taste, very pleasing. Most of the utility comes from two observations: first, there is a collection of subrings, depending only on the Grothendieck ring of a Severi-Brauer variety X, of the Chow ring of X that behave nicely with regards to the geometry of a Severi-Brauer variety (e.g. with respect to the structure of the Severi-Brauer subvarieties, or with respect to the twisted Veronese or Segre embeddings); second, these subrings can be shown, in some cases, to satisfy almost all of the functorality that the Chow ring itself satisfies (e.g. there are pullbacks, and occasionally pushforwards that, when they exist, satisfy the projection formula).

One can also generalize the situation above a bit. It's known from [Kar98] that if the reduced behavior of an algebra A, assuming A has p-power dimension, decreases by 1 at every step, or if the algebra A has $p = 2$ and at the second to last step goes down by 2, then the Severi-Brauer variety $X = \mathrm{SB}(A)$ can't have torsion in $\mathrm{CH}^2(X)$. The work in [Kar98] also shows that for any other possible reduced behavior, there is an algebra A with this reduced behavior and such that the group $\mathrm{CH}^2(\mathrm{SB}(A))$ contains nontrivial torsion. This leads one to the following:

Question: For which reduced behaviors R can one construct an algebra A such that: the reduced behavior of A is R, and the Chow group of codimension 2 cycles of the Severi-Brauer variety of A is torsion free?

Now we give a rough sketch of the structure of the paper. In section 2, we discuss preliminaries on the structure of the Chow ring of a Severi-Brauer variety. Most of this material was developed in [KM19] but it is recalled here for convenience. In section 3 we discuss functorality of a collection of rings $\mathrm{CT}(i;-)$ and prove that, in low codimensions the group summands of these rings have pushforwards. We also introduce a group $\mathrm{Q}(-)$ that has been implicitly used in [Kar17, KM19] to compute Chow rings of particular Severi-Brauer varieties and we study this group in low codimension as well. When pushforwards on $\mathrm{CT}(1;-)$ exist, they then also exist for $\mathrm{Q}(-)$; these pushforwards are our main computational tool throughout this paper. Section 4 concludes with a sample of how these tools could be used, e.g. for applications to torsion in $\mathrm{CH}^2(-)$.

2. Preliminaries

Let A be a central simple algebra (over k). Over an algebraic closure, say $\overline{k}$, there is an isomorphism $A_{\overline{k}} = A \otimes_k \overline{k} \cong M_n(\overline{k})$ with the ring of $n \times n$ square matrices. The dimension of A is then a square, and we denote by $\deg(A)$ its square root; the number $\deg(A)$ is called the degree of A.

We write $X = \mathrm{SB}(A)$ for the associated Severi-Brauer variety of A. By definition, X is the closed subvariety of the Grassmannian $\mathrm{Gr}(\deg(A), A)$ of hyperplanes of dimension $\deg(A)$ inside of A whose F-points, over any field extension F of k, are the left ideals of A_F. We write

$$\varphi_A : X \to \mathrm{Gr}(\deg(A), A)$$

to denote the canonical inclusion.

The Grassmannian comes equipped with a universal subbundle, denoted $\mathcal{S}_{Gr}$, of rank $\deg(A)$. As a vector bundle, the fiber over an R-point x, for a finite type k-algebra R, of $\mathcal{S}_{Gr}$ is the projective R-module summand of $A \otimes_k R$ defining x. The pullback $\zeta_X := \varphi_A^*(\mathcal{S}_{Gr})$ is called the *tautological bundle over* X or, when considering the associated sheaf of ζ_X, the *tautological sheaf over* X.

By construction, the sheaf ζ_X is a left module under the constant sheaf $\underline{A}$ of A on X. For any integer $i \geq 0$, we pick a simple right $A^{\otimes i}$-module M_i. From now on we use the notation $\zeta_X(i) := \underline{M_i} \otimes_{\underline{A}^{\otimes i}} \zeta_X^{\otimes i}$ to denote the given tensor product. Since the isomorphism class of M_i is uniquely determined by the theory of central simple algebras, see [Jac89, Chapter 4], it follows that $\zeta_X(i)$ is uniquely defined, up to an isomorphism.

For a given i, the sheaf $\zeta_X(i)$ is locally free of rank the index of $A^{\otimes i}$, i.e. $\mathrm{rk}(\zeta_X(i)) = \mathrm{ind}(A^{\otimes i}) = \dim(M_i)$. The collection of such sheaves completely determines the Grothendieck ring $K(X)$ of X.

Theorem 2.1 ([Qui73, §8, Theorem 4.1])**.** *The group homomorphism*

$$\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X),$$

sending the class of a right $A^{\otimes i}$-module M to $\underline{M} \otimes_{\underline{A}^{\otimes i}} \zeta_X^{\otimes i}$, is an isomorphism.

In particular, it follows from Theorem 2.1 that $K(X)$ is additively generated by $\zeta_X(i)$ as i varies in the interval $0 \leq i \leq \deg(A) - 1$. Moreover, for any field extension F/k, the homomorphism of Theorem 2.1 commutes with the pullback to F, the induced morphism $K(X) \to K(X_F)$ is injective, and $K(X)$ can be identified with a subring of $K(X_F)$. When F is a field that splits A, meaning that there is an isomorphism $A_F \cong M_n(F)$, then there is the following well-known description of this situation:

Theorem 2.2. *Let ξ denote the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$. Then there is a ring isomorphism*

$$\mathbb{Z}[x]/(1-x)^n \xrightarrow{\sim} K(X_F)$$

sending x to ξ.

Under this isomorphism, the class of $\zeta_X(i)$ is sent to $\mathrm{ind}(A^{\otimes i})x^i$ *and $K(X)$ identifies with the subring of $\mathbb{Z}[x]/(1-x)^n$ generated by these classes.*

Due to the description of $K(X)$ given by Theorem 2.2, the sequence of integers $\mathrm{ind}(A^{\otimes i})$ as i increases turns out to be an important invariant when studying the intersection theory of X. But, although it's sufficient, one doesn't need to consider all integers i to have a complete description of $K(X)$; there are two more observations that one can make to drastically simplify the situation. For our purposes, it will suffice to assume $\mathrm{ind}(A) = p^r$ is a power of a prime p. In the rest of this section we make this assumption.

The first observation is that the indices $\mathrm{ind}(A^{\otimes i})$ depend only on the p-adic valuation, $v_p(i)$, of the integer i so, to know all of these indices, it suffices to know only the indices of the prime powers $\mathrm{ind}(A^{\otimes p^i})$. This information is contained in the reduced behavior of A, defined as the sequence

$$r\mathcal{B}eh(A) = \left(v_p\mathrm{ind}(A^{\otimes p^i})\right)_{i=0}^{m}$$

where $m = v_p\exp(A)$ is the p-adic valuation of the exponent of A, i.e. the p-adic valuation of the order of A in the Brauer group of k. Since for any $1 \le i \le m$ there is an inequality

$$v_p\mathrm{ind}(A^{\otimes p^i}) \le v_p\mathrm{ind}(A^{\otimes p^{i-1}}) - 1, \tag{in}$$

by [Kar98, Lemma 3.10], it follows that the reduced behavior is a strictly decreasing sequence ending in 0. We note also, again by [Kar98, Lemma 3.10], that for any prime p and any strictly decreasing sequence S ending in 0, there is a central simple algebra B with p-primary index having $r\mathcal{B}eh(B) = S$.

The second observation is that one only needs to consider a finite subset of the reduced behavior to completely determine $K(X)$. Let

$$S_X = \{i : v_p\mathrm{ind}(A^{\otimes p^i}) < v_p\mathrm{ind}(A^{\otimes p^{i-1}}) - 1\}$$

be those integers i between 1 and m where the inequality (in) fails to be an equality. In [KM19], the cardinality $\#S_X$ is called the level of A; here we also call $\#S_X$ the level of X. The relevance of S_X is apparent from the following lemma.

Lemma 2.3 ([KM19, Lemma A.6])**.** *The ring $K(X)$ is generated, as a λ-ring, by the classes of the vector bundles of the set $\{\zeta_X(p^i)\}_i$ where i is an index for $S_X \cup \{0\}$.*

Aside from providing a generating set for $K(X)$, Lemma 2.3 also gives valuable information on the structure of the Chern subring of the Chow ring of X. Since Chern classes in λ-operations of an element x are polynomials in Chern classes of x, Lemma 2.3 implies that the Chern subring of $\mathrm{CH}(X)$ is generated by Chern classes of the same bundles. From now on we'll use the following notation:

Definition 2.4. The notation $\mathrm{CT}(i_1, \ldots, i_j; X)$ will denote the graded subring of the Chow ring $\mathrm{CH}(X)$ generated by the Chern classes of $\zeta_X(i_1), \ldots, \zeta_X(i_j)$.

Only the rings generated by Chern classes of one vector bundle are likely to appear. And among these rings we focus on $\mathrm{CT}(1; X)$ the most since this can often be used to get information on the rings $\mathrm{CT}(i; X)$. That these rings are interesting comes from the following description of their generators.

Proposition 2.5 ([KM19, Proposition A.8])**.** *For any $i > 0$, the ring $\mathrm{CT}(i; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Moreover, for $0 \le j < \deg(A)$ the degree j summand $\mathrm{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ is additively generated by the element*

$$\tau_i(j) := c_{p^v}(\zeta_X(i))^{s_0} c_{s_1}(\zeta_X(i))$$

where p^v is the largest power of p dividing $\mathrm{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \le s_1 < p^v$.

When there's possible ambiguity for where these classes are defined, or if confusion could occur, we will include a superscript like $\tau_i^X(j)$ to mean these classes are defined inside $\mathrm{CT}(i; X) \otimes \mathbb{Z}_{(p)}$.

3. Chow groups of Severi-Brauer varieties

Throughout this section we work with a central simple algebra A with $\mathrm{ind}(A) = p^n$, for some fixed prime p and some $n \geq 0$, and we write $X = \mathrm{SB}(A)$ for the associated Severi-Brauer variety. We keep all other notations from Section 2.

3.1. **Functorality of** $\mathrm{CT}(1; -)$**.** Let F/k be a field extension and write $\pi_{F/k} : X_F \to X$ for the projection from F. Since $\pi_{F/k}$ is flat, and since flat pullback commutes with Chern classes, the map $\pi_{F/k}^* : \mathrm{CH}(X) \to \mathrm{CH}(X_F)$ induces a map

$$\pi_{F/k}^* : \mathrm{CT}(i; X) \to \mathrm{CT}(i; X_F)$$

for each $i \geq 1$; this induced morphism will also be called the pullback along $\pi_{F/k}$ and written as $\pi_{F/k}^*$ by abuse of notation.

Our main interest is when F/k is a finite extension. In this case one sometimes also gets an induced pushforward on the groups $\mathrm{CT}^j(i; -)$ for various i, j. Our goal at the moment is to describe properties of such a pushforward when it exists. Afterwards, we prove that these pushforwards exist in cases we will be interested in, i.e. for all $i \geq 0$ and for $j = 0, 1, 2$.

Lemma 3.1. *Write D_A for the underlying division algebra of A, and set $Y = \mathrm{SB}(D_A)$ to be the corresponding Severi-Brauer variety. Fix a finite field extension F/k and write $\pi_{F/k} : X_F \to X$ for the projection. Suppose that, for any fixed $0 \leq j \leq \dim(Y)$ (resp. for every $j \geq 0$) the composition*

$$\mathrm{CT}^j(1; Y_F) \otimes \mathbb{Z}_{(p)} \subset \mathrm{CH}^j(Y_F) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(Y) \otimes \mathbb{Z}_{(p)}$$

has image contained in $\mathrm{CT}^j(1; Y) \otimes \mathbb{Z}_{(p)}$. Then the same is true for X: for this fixed j (resp. for every $j \geq 0$) the composition

$$\mathrm{CT}^j(1; X_F) \otimes \mathbb{Z}_{(p)} \subset \mathrm{CH}^j(X_F) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X) \otimes \mathbb{Z}_{(p)}$$

has image contained in $\mathrm{CT}^j(1; X) \otimes \mathbb{Z}_{(p)}$.

Proof. In this case, there is a closed immersion $Y \to X$ that becomes the linear inclusion of projective spaces over an algebraic closure of k. This immersion then induces a commuting diagram made of pushforward maps and Gysin pullbacks

$$\begin{array}{ccc} \mathrm{CH}(X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{CH}(Y_F) \otimes \mathbb{Z}_{(p)} \\ \downarrow{\scriptstyle \pi_{F/k,*}} & & \downarrow{\scriptstyle \pi_{F/k,*}} \\ \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{CH}(Y) \otimes \mathbb{Z}_{(p)} \end{array}$$

In degrees where both $\mathrm{CH}(X)$ and $\mathrm{CH}(Y)$ are nonzero the Gysin pullback is an isomorphism and this induces an isomorphism $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CT}(1; Y) \otimes \mathbb{Z}_{(p)}$ in the same degrees. This proves the lemma in all degrees less than $p^n - 1$ by going around the diagram above.

To complete the proof when these maps are known to exist for Y and for all $j \geq 0$ one uses the projection formula. For any $j \geq p^n$ consider the class $\tau_1(j)$ of $\mathrm{CT}^j(1; X_F) \otimes \mathbb{Z}_{(p)}$ from Proposition 2.5; this class is an additive generator for $\mathrm{CT}^j(1; X_F) \otimes \mathbb{Z}_{(p)}$ so it is sufficient to prove that $\pi_{F/k,*}(\tau_1(j))$ is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$.

One has

$$\pi_{F/k,*}(\tau_1(j)) = \pi_{F/k,*}(c_{p^v}(\zeta_{X_F}(1))^{s_0} c_{s_1}(\zeta_{X_F}(1)))$$

where $j = p^v s_0 + s_1$ and $0 \leq s_1 < p^v$ and $v = v_p \mathrm{ind}(A_F)$. Since $p^n \leq j$, it follows that $p^{n-v} \leq s_0$. Because of the commutativity of Chern classes and restriction over field extensions, we have that

$$c_{p^v}(\zeta_{X_F}(1))^{p^{n-v}} = \pi_{F/k}^*(c_{p^n}(\zeta_X(1))).$$

Indeed, the Chern classes are computable

$$\pi_{F/k}^* c_t(\zeta_X(1)) = c_t(\zeta_{X_F}(1))^{p^{n-v}} = \big(1 + c_1(\zeta_{X_F}(1))t + \cdots + c_{p^v}(\zeta_{X_F}(1))t^{p^v}\big)^{p^{n-v}}$$

and, after expanding, one gets the claim as the coefficient of t^{p^n} in this polynomial. Then by the projection formula (in $\mathrm{CH}(X_F) \otimes \mathbb{Z}_{(p)}$) it follows

$$\begin{aligned}\pi_{F/k,*}(\tau_1(j)) &= \pi_{F/k,*}(\pi_{F/k}^*(c_{p^n}(\zeta_X(1)))c_{p^v}(\zeta_{X_F}(1))^{s_0-p^{n-v}}c_{s_1}(\zeta_{X_F}(1))) \\ &= c_{p^n}(\zeta_X(1))\pi_{F/k,*}(c_{p^v}(\zeta_{X_F}(1))^{s_0-p^{n-v}}c_{s_1}(\zeta_{X_F}(1))).\end{aligned}$$

Since $c_{p^n}(\zeta_X(1))$ is contained in $\mathrm{CT}(1; X)$ by definition, the lemma is proved by an inductive argument on the degree j of $\tau_1(j)$. □

Lemma 3.2. *Fix a finite field extension F/k and write $\pi_{F/k} : X_F \to X$ for the projection. Suppose that, for any fixed $j \geq 0$, the composition*

$$\mathrm{CT}^j(1; X_F) \otimes \mathbb{Z}_{(p)} \subset \mathrm{CH}^j(X_F) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X) \otimes \mathbb{Z}_{(p)}$$

has image contained in $\mathrm{CT}^j(1; X) \otimes \mathbb{Z}_{(p)}$. Fix an integer $i > 1$, let $Y = \mathrm{SB}(A^{\otimes i})$, and write $\pi_{F/k} : Y_F \to Y$ for the projection by abuse of notation. Suppose, in addition, that the composition

$$\mathrm{CT}^j(1; Y_F) \otimes \mathbb{Z}_{(p)} \subset \mathrm{CH}^j(Y_F) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(Y) \otimes \mathbb{Z}_{(p)}$$

has image contained in $\mathrm{CT}^j(i; Y) \otimes \mathbb{Z}_{(p)}$.

Then, for any $i > 1$, the same is true for $\mathrm{CT}^j(i; X)$: for this fixed $j \geq 0$ the composition

$$\mathrm{CT}^j(i; X_F) \otimes \mathbb{Z}_{(p)} \subset \mathrm{CH}^j(X_F) \otimes \mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X) \otimes \mathbb{Z}_{(p)}$$

has image contained in $\mathrm{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$.

Proof. Consider the twisted Veronese embedding

$$\rho : X \to X \times \cdots \times X = X^{\times i} \to Y$$

which factors via the diagonal map $X \to X^{\times i}$ and the closed immersion $X^{\times i} \to Y$ defined on R-points, for any finite type k-algebra R, as the map taking an i-tuple of left ideals $(J_1, ..., J_i)$ to the tensor product $J_1 \otimes \cdots \otimes J_i$. Note that over an algebraic closure of k the map ρ can be canonically identified with the usual Veronese embedding of projective space so that $\rho^*\zeta_Y(1) = \zeta_X(i)$, see [KM19, Lemma A.9].

Let $\rho_F : X_F \to Y_F$ be the morphism induced by ρ after extending scalars to F. Then ρ fits into a Cartesian diagram

$$\begin{array}{ccc} X_F & \xrightarrow{\rho_F} & Y_F \\ \big\downarrow{\scriptstyle \pi_{F/k}} & & \big\downarrow{\scriptstyle \pi_{F/k}} \\ X & \xrightarrow{\rho} & Y \end{array}$$

with the right vertical map the projection from F, also written $\pi_{F/k}$ by abuse of notation. By [EKM08, Corollary 55.4], the diagram induces a commutativity on Chow groups between the push-forwards of the vertical maps and the Gysin pullbacks along ρ, ρ_F:

$$\pi_{F/k,*} \circ \rho_F^* = \rho^* \circ \pi_{F/k,*}.$$

By construction, this induces a commuting diagram as below.

$$\begin{array}{ccc} \mathrm{CT}(1;Y_F)\otimes\mathbb{Z}_{(p)} & \xrightarrow{\rho_F^*} & \mathrm{CT}(i;X_F)\otimes\mathbb{Z}_{(p)} \\ \downarrow{\scriptstyle \pi_{F/k,*}} & & \downarrow{\scriptstyle \pi_{F/k,*}} \\ \mathrm{CT}(1;Y)\otimes\mathbb{Z}_{(p)} & \xrightarrow{\rho^*} & \mathrm{CH}(X)\otimes\mathbb{Z}_{(p)} \end{array}$$

As the horizontal arrows surject onto $\mathrm{CT}(i;-)\otimes\mathbb{Z}_{(p)}$, one can conclude with a diagram chase. □

Remark 3.3. Lemma 3.1 (and Lemma 3.2) holds with integral coefficients also. But it's less convenient to prove, since one needs to consider the given Chern classes and powers of the first Chern class.

Lemma 3.4. *Let F/k be a finite field extension and $\pi_{F/k}: X_F \to X$ the projection. Suppose that the composition (resp. this composition with $\mathbb{Z}_{(p)}$-coefficients)*

$$\mathrm{CT}^j(i;X_F)\subset \mathrm{CH}^j(X_F)\xrightarrow{\pi_{F/k,*}}\mathrm{CH}^j(X)$$

has image contained in $\mathrm{CT}^j(i;X)$ (resp. $\mathrm{CT}^j(i;X)\otimes\mathbb{Z}_{(p)}$). Then the projection formula holds for $\pi_{F/k,*},\pi^*_{F/k}$ and the compositions

$$\pi^*_{F/k}\circ\pi_{F/k,*}\quad \textit{and}\quad \pi_{F/k,*}\circ\pi^*_{F/k}$$

are both multiplication by $[F:k]$.

Proof. The projection formula holds as these maps are induced by maps where the projection formula is known to hold. To see the claim on compositions one can note that, in any degree j, the groups $\mathrm{CT}^j(i;X)$ and $\mathrm{CT}^j(i;X_F)$ are isomorphic with $\mathbb{Z}$. Then, by the projection formula, $\pi_{F/k,*}\circ\pi^*_{F/k}=[F:k]$ so the same must be true for the other composition. □

Lemma 3.5. *Let F/k be a finite extension splitting A and $\pi_{F/k}: X_F\to X$ the projection. Then the composition*

$$\mathrm{CT}^j(i;X_F)\subset \mathrm{CH}^j(X_F)\xrightarrow{\pi_{F/k,*}}\mathrm{CH}^j(X)$$

has image contained in $\mathrm{CT}^j(i;X)$ for any $j\geq 0$ and for any $i\geq 1$.

Proof. This is proved, when A is a division algebra and $i=1$, in [Kar17, Proposition 3.5]. The general case follows from Lemmas 3.1 and 3.2. □

Proposition 3.6. *Let F/k be a finite field extension and $\pi_{F/k}: X_F\to X$ the projection. Then the composition*

$$\mathrm{CT}^j(i;X_F)\subset \mathrm{CH}(X_F)\xrightarrow{\pi_{F/k,*}}\mathrm{CH}(X)$$

has image in $\mathrm{CT}^j(i;X)$ when $j=0,1$ and for any $i\geq 1$.

And, if one localizes at the prime ideal generated by p, the same holds in codimension 2. The composition

$$\mathrm{CT}^2(i;X_F)\otimes\mathbb{Z}_{(p)}\subset \mathrm{CH}(X_F)\otimes\mathbb{Z}_{(p)}\xrightarrow{\pi_{F/k,*}}\mathrm{CH}(X)\otimes\mathbb{Z}_{(p)}$$

has image in $\mathrm{CT}^2(i;X)\otimes\mathbb{Z}_{(p)}$ for any $i\geq 1$.

Proof. The claim for $j=0$ is trivial. The claim for $j=1$ is a remark on the Picard group. More precisely, since $\mathrm{Pic}(X_F)$ is torsion free for any finite extension F/k, we can prove the claim with a computation of Chern classes. One has

$$\pi^*_{F/k}(c_1(\zeta_X(i)))=\frac{\mathrm{ind}(A^{\otimes i})}{\mathrm{ind}(A_F^{\otimes i})}c_1(\zeta_{X_F}(i))$$

and by the projection formula this means

$$\frac{\operatorname{ind}(A^{\otimes i})}{\operatorname{ind}(A_F^{\otimes i})}\pi_{F/k,*}(c_1(\zeta_{X_F}(i))) = [F:k]c_1(\zeta_X(i)).$$

Then one can divide by the coefficient on the left to get some integer on the multiple of $c_1(\zeta_X(i))$ on the right.

Now we turn to proving that there is a morphism

$$\mathrm{CT}^2(i;X_F)\otimes\mathbb{Z}_{(p)} \xrightarrow{\pi_{F/k,*}} \mathrm{CT}^2(i;X)\otimes\mathbb{Z}_{(p)}.$$

By Lemma 3.2 it suffices to show the case $i=1$. The proof takes some setting-up.

We're going to use the Grothendieck-Riemann-Roch without denominators, [Ful98, Example 15.3.6]. We write $\mathrm{gr}_\tau G(X) = \bigoplus_{i\geq 0}\tau^i/\tau^{i+1}$ for the graded ring associated to the topological filtration τ^* on the Grothendieck ring of coherent sheaves on X. The Grothendieck-Riemann-Roch without denominators gives maps

$$\varphi_X^2 : \mathrm{CH}^2(X)\to \mathrm{gr}_\tau^2 G(X) \quad\text{and}\quad c_2:\mathrm{gr}_\tau^2G(X)\to\mathrm{CH}^2(X)$$

such that the compositions $\varphi_X^2\circ c_2$ and $c_2\circ\varphi_X^2$ are both multiplication by -1. The maps φ_X^2 (with different X) are functorial in the sense that they take Chern classes to Chern classes and commute with pushforwards. The same holds after localizing at (p) so it suffices to prove the claim by working with the images $C=\varphi_X^2(\mathrm{CT}^2(1;X)\otimes\mathbb{Z}_{(p)})$ and $C'=\varphi_{X_F}^2(\mathrm{CT}^2(1;X_F)\otimes\mathbb{Z}_{(p)})$ in $\mathrm{gr}_\tau^2G(X)\otimes\mathbb{Z}_{(p)}$ and $\mathrm{gr}_\tau^2G(X_F)\otimes\mathbb{Z}_{(p)}$ respectively.

We can assume A_F is not split, because of Lemma 3.5. We set $\operatorname{ind}(A)=p^n$ and $\operatorname{ind}(A_F)=p^{n-v}$ below. By our description from Proposition 2.5, the generator $\varphi_{X_F}^2\left(\tau_1^{X_F}(2)\right)$ in C' is the class of the K-theoretic Chern class $c_2^K(\zeta_{X_F}(1))$ of $G(X_F)\otimes\mathbb{Z}_{(p)}$. Using the projection formula, we're going to show $\pi_{F/k,*}(c_2^K(\zeta_{X_F}(1)))$ is a multiple of $c_2^K(\zeta_X(1))$ in $G(X)\otimes\mathbb{Z}_{(p)}$ which will complete the proof. First we show something similar for $\pi_{F/k}^*$.

Let L be a splitting field for X containing F and let $\pi_{L/F}:X_L\to X_F$ be the projection. The following diagram commutes and all of the arrows in it are injective.

$$\begin{array}{ccc} G(X_F)\otimes\mathbb{Z}_{(p)} & \xrightarrow{\pi_{L/F}^*} & G(X_L)\otimes\mathbb{Z}_{(p)} \\ \uparrow{\scriptstyle \pi_{F/k}^*} & \nearrow{\scriptstyle \pi_{L/k}^*} & \\ G(X)\otimes\mathbb{Z}_{(p)} & & \end{array}$$

Using the notation of Theorem 2.2, and writing $\xi^\vee$ for the dual class, we have equalities

$$\binom{p^n}{2}(1-\xi^\vee)^2 = \pi_{L/k}^*\left(c_2^K(\zeta_X(1))\right) = \pi_{L/F}^*(\pi_{F/k}^*(c_2^K(\zeta_X(1))))$$

and

$$\binom{p^n}{2}(1-\xi^\vee)^2 = \frac{\binom{p^n}{2}}{\binom{p^{n-v}}{2}}\binom{p^{n-v}}{2}(1-\xi^\vee)^2 = \pi_{L/F}^*\left(\frac{\binom{p^n}{2}}{\binom{p^{n-v}}{2}}c_2^K(\zeta_{X_F}(1))\right)$$

implying

$$\pi_{F/k}^*(c_2^K(\zeta_X(1))) = \frac{\binom{p^n}{2}}{\binom{p^{n-v}}{2}}c_2^K(\zeta_{X_F}(1))$$

by the injectivity of $\pi_{L/F}^*$.

Thus, in $G(X)\otimes\mathbb{Z}_{(p)}$ we have

$$[F:k]c_2^K(\zeta_X(1)) = \pi_{F/k,*}\left(\pi_{F/k}^*\left(c_2^K(\zeta_X(1))\right)\right) = \frac{\binom{p^n}{2}}{\binom{p^{n-v}}{2}}\pi_{F/k,*}(c_2^K(\zeta_{X_F}(1))).$$

But, since $v = v_p\left(\binom{p^n}{2}/\binom{p^{n-v}}{2}\right) \leq v_p([F:k])$ and $G(X)\otimes\mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module, it follows that one can divide by the coefficient of the final line of this equation. □

3.2. **Generalities on** $\mathrm{Q}(X)$**.** Consider the inclusion of $\mathrm{CT}(1;X)$ into $\mathrm{CH}(X)$. We denote the group cokernel of this inclusion by $\mathrm{Q}(X)$. This is a graded group, with degree j summand the quotient

$$\mathrm{Q}^j(X) := \mathrm{CH}^j(X)/\mathrm{CT}^j(1;X).$$

For any field extension F/k the pullback $\pi_{F/k}^*$ induces a morphism

$$\pi_{F/k}^* : \mathrm{Q}^j(X) \to \mathrm{Q}^j(X_F).$$

If F/k is a finite field extension, and if the composition

$$\mathrm{CT}^j(1;X_F) \subset \mathrm{CH}^j(X_F) \xrightarrow{\pi_{F/k,*}} \mathrm{CH}^j(X)$$

has image in $\mathrm{CT}^j(1;X)$ then there is also a pushforward morphism

$$\pi_{F/k,*} : \mathrm{Q}^j(X_F) \to \mathrm{Q}^j(X).$$

When these pushforwards exist the composition $\pi_{F/k,*}\circ\pi_{F/k}^*$ is multiplication by $[F:k]$ on $\mathrm{Q}^j(X)$.

Choosing F to be a maximal subfield of the underlying division algebra of A, and using Lemma 3.5, it follows that the degree $p^n = [F:k]$ annihilates the group $\mathrm{Q}(X)$. Thus, $\mathrm{Q}(X)$ is a p-torsion group and can also be realized as the cokernel of the inclusion $\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)} \to \mathrm{CH}(X)\otimes\mathbb{Z}_{(p)}$.

In low degrees $j = 0,1,2$, when the group $\mathrm{CH}^j(X)$ is generated by Chern classes, one can describe $\mathrm{Q}^j(X)$ fairly explicitly. Recall that $S_X = \{i : v_p\mathrm{rk}(\zeta_X(p^i)) < v_p\mathrm{rk}(\zeta_X(p^{i-1})) - 1\}$ is our notation for the set considered in Section 2 depending on the reduced behavior of A. We can consider S_X as an ordered set of integers $S_X = \{i_1,\dots,i_k\}$ with $i_1 < \cdots < i_k$. We write $n_r = v_p\mathrm{rk}(\zeta_X(p^r))$ for any $r \geq 0$ in the following.

Proposition 3.7. *In the notation above, there are isomorphisms*

$$\mathrm{Q}^j(X) \cong \begin{cases} 0 & \text{if} \quad j=0 \\ \exp(A)\mathbb{Z}/\mathrm{ind}(A)\mathbb{Z} & \text{if} \quad j=1 \\ \langle \tau_{p^{i_1}}(2),\dots,\tau_{p^{i_k}}(2)\rangle & \text{if} \quad j=2 \end{cases}$$

In particular, the group $\mathrm{Q}^1(X)$ is generated by the image of $\tau_{p^{i_k}}(1)$; the group $\mathrm{Q}^2(X)$ is generated by the images of $\tau_{p^i}(2)$ for i varying over the elements of S_X.

The orders of the (images of the) $\tau_{p^r}(2)$ are bounded above by p^{n-n_r-r} if $n_r \neq 0$ and by $p^{n-r-v_p(2)}$ if $n_r = 0$. For each pair r,s with $r > s \geq 0$ there are relations

$$\beta_{r,s}\tau_{p^s}(2) = \alpha_2^{r,s}\tau_{p^r}(2)$$

for some $\alpha_2^{r,s},\beta_{r,s}$ in $\mathbb{Z}_{(p)}$ that have p-adic valuation

$$v_p(\alpha_2^{r,s}) = \begin{cases} n_s - n_r - (r-s) & \text{if} \quad n_r \neq 0 \\ n_s - (r-s) - v_p(2) & \text{if} \quad n_r = 0 \end{cases} \quad \text{and} \quad v_p(\beta_{r,s}) = r-s.$$

Remark 3.8. There could possibly be further relations in the group $Q^2(X)$. For example, if p is an odd prime, if A is a division algebra of index p^2, exponent p and if $\mathrm{CH}(X)$ is generated by Chern classes then $\#Q^2(X) = p$. But, if A is decomposable into a tensor product of two degree p algebras, then $\#Q^2(X) = 0$.

Proof of Proposition 3.7. The statements for $Q^0(X)$ and $Q^1(X)$ are well-known. The content of the lemma is mostly in the statement for $Q^2(X)$.

To prove the statement we are going to introduce temporary notation for a set of generators of $Q^2(X)$. Since $\mathrm{CH}^2(X)$ is generated by Chern classes, and all Chern classes from $K(X)$ are classes in $\zeta_X(p^i)$ with $i \in S_X \cup \{0\}$ by Lemma 2.3, it's clear that $Q^2(X)$ is additively generated by monomials

$$x_{a,b}(i,j) := \tau_{p^j}(a)\tau_{p^i}(b)$$

where $i, j \in S_X \cup \{0\}$ is a pair of integers, and $a, b \geq 0$ is another pair of integers with $a + b = 2$.

We work in cases from here. The class $x_{a,b}(0,0)$ can be omitted for any a, b since this element is trivial in $Q^2(X)$. We can also eliminate the elements $x_{a,b}(1,1)$ because $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$ is an infinite cyclic group generated by $\tau_{p^{i_k}}(1)$. More precisely, it follows that any other first Chern class is a multiple of this one and there are equalities

$$x_{a,b}(1,1) = m_{a,b}\tau_{p^{i_k}}(1)^2$$

in $\mathrm{CH}^2(X) \otimes \mathbb{Z}_{(p)}$ for some $m_{a,b} \in \mathbb{Z}_{(p)}$. But, since $\mathrm{CT}^2(p^{i_k}; X) \otimes \mathbb{Z}_{(p)}$ is additively generated by $\tau_{p^{i_k}}(2)$ by Proposition 2.5, it follows there is an equality

$$x_{a,b}(1,1) = m'_{a,b}\tau_{p^{i_k}}(2)$$

in $\mathrm{CH}^2(X) \otimes \mathbb{Z}_{(p)}$ for some $m'_{a,b} \in \mathbb{Z}_{(p)}$. The same holds then in $Q^2(X)$.

Now consider the remaining classes $x_{0,2}(i,j) = \tau_{p^j}(2)$ for varying $i, j \in S_X$ (or, equivalently, the classes $x_{2,0}(i,j)$). That these classes are bounded above by the specified power of p is a consequence of [KM19, Corollary A.13] (see also Lemma 3.9 below). To get the relations, we are going to show that for every $r > s$ there is an equality

$$\beta_{r,s}\tau_{p^s}(2) = \alpha_2^{r,s}\tau_{p^r}(2)$$

in $\mathrm{CH}^2(X) \otimes \mathbb{Z}_{(p)}$ for some $\alpha_2^{r,s}, \beta_{r,s} \in \mathbb{Z}_{(p)}$ with the given properties. This is done in the following lemma and in the subsequent corollary. □

Lemma 3.9. *Pick an integer r with $0 \leq r \leq v_p\exp(A)$ and let i be an integer bounded like $0 \leq i < \mathrm{ind}(A) = p^n$. Then for all integers s with $0 \leq s \leq r$, there exists a number $\alpha_i^{r,s}$ of $\mathbb{Z}_{(p)}$ such that $\alpha_i^{r,s}\tau_{p^r}(i)$ is contained in $\mathrm{CT}(p^s; X) \otimes \mathbb{Z}_{(p)}$.*

Moreover, the p-adic valuation of the $\alpha_i^{r,s}$ we find is equal

$$v_p(\alpha_i^{r,s}) = \begin{cases} n_s - n_r - (r-s) & \text{if } \ 1 \leq i \leq p^{n_r} \\ n_s - n_r - (r-s) - \lfloor \log_p(i/p^{n_r}) \rfloor & \text{if } \ p^{n_r} < i \leq p^{n_s-(r-s)} \\ 0 & \text{if } \ p^{n_s-(r-s)} < i \leq p^{n_s} - 1. \end{cases}$$

Proof. When $s = 0$, and A is a division algebra this is proved in [KM19, Corollary A.13]. To get the claim in the case $s = 0$ and A is an arbitrary central simple algebra one can consider the Gysin pullback of $Y = \mathrm{SB}(D_A) \to X$, where D_A is the underlying division algebra of A, and note that $\mathrm{CT}(i; X) = \mathrm{CT}(i; Y)$ in these degrees.

In the case $s > 0$, write $Z = \mathrm{SB}(A^{\otimes p^s})$ and consider the composition

$$X \to X \times \cdots \times X = X^{\times p^s} \to Z.$$

The pullback along this map $K(Z) \to K(X)$ sends $\zeta_Z(1)$ to $\zeta_X(p^s)$. In this way one gets a surjection

$$\rho : \mathrm{CT}(1; Z) \twoheadrightarrow \mathrm{CT}(p^s; X).$$

Recall that, in order to distinguish the classes we consider, we are writing $\tau_i^Z(j)$ for the class in $\mathrm{CT}^j(i;Z)\otimes\mathbb{Z}_{(p)}$ and $\tau_i^X(j)$ for the class in $\mathrm{CT}^j(i;X)\otimes\mathbb{Z}_{(p)}$.

Then, it follows since $\alpha_i^{r-s,0}\tau^Z_{p^{r-s}}(i)$ is an element of $\mathrm{CT}^i(1;Z)\otimes\mathbb{Z}_{(p)}$ that

$$\rho\left(\alpha_i^{r-s,0}\tau^Z_{p^{r-s}}(i)\right)=\alpha_i^{r-s,0}\rho\left(\tau^Z_{p^{r-s}}(i)\right)=\alpha_i^{r-s,0}\tau^X_{p^r}(i)$$

is an element of $\mathrm{CT}(p^s;X)\otimes\mathbb{Z}_{(p)}$. Finally, writing $n'_j=v_p\mathrm{rk}(\zeta_Z(p^j))$ and recalling $n_j=v_p\mathrm{rk}(\zeta_X(p^j))$, we find

$$v_p(\alpha_i^{r-s,0})=\begin{cases} n'_0-n'_{r-s}-(r-s) & \text{if } \quad 1\le i\le p^{n'_{r-s}}\\ n'_0-n'_{r-s}-(r-s)-\lfloor\log_p(i/p^{n'_{r-s}})\rfloor & \text{if } \quad p^{n'_{r-s}}<i\le p^{n'_0-(r-s)}\\ 0 & \text{if } \quad p^{n'_0-(r-s)}<i\le p^{n'_0}-1.\end{cases}$$

The claim then follows since $n'_0=n_s$, and $n'_{r-s}=n_r$ □

Corollary 3.10. *Keep notation as above. Then, for all $0\le s<r$, there are equalities*

$$\beta_{r,s}\tau_{p^s}(2)=\alpha_2^{r,s}\tau_{p^r}(2)$$

in $\mathrm{CH}^2(X)\otimes\mathbb{Z}_{(p)}$ for an element $\beta_{r,s}$ of $\mathbb{Z}_{(p)}$ with $v_p(\beta_{r,s})=r-s$.

Proof. It follows from Lemma 3.9 that there is some equality

$$\beta_{r,s}\tau_{p^s}(2)=\alpha_2^{r,s}\tau_{p^r}(2)$$

for an element $\beta_{r,s}$ of $\mathbb{Z}_{(p)}$. To get the result, it suffices to compute the p-adic valuations of $\tau_{p^s}(2),\tau_{p^r}(2)$, and $\alpha_2^{r,s}$ after they are pulled back to an algebraic closure. This is left to the reader. □

4. Decomposability

In this section our goal is to describe a new method for checking torsion freeness of $\mathrm{CH}^2(X)$ where X is the Severi-Brauer variety of a central simple algebra A satisfying: A is a division algebra, $\mathrm{ind}(A)=p^n$ is a power of an odd prime p and, over a field extension F/k, the group $\mathrm{CH}^2(X_F)$ is torsion free.

Our main result in this regard is that, under some assumptions on the decomposability of A, the group $\mathrm{CH}^2(X)$ is torsion free. This extends some of the previously known examples from [Kar98, Kar96]. More precisely, we show:

Theorem 4.1. *Keep notations as above. Assume further that A decomposes into a tensor product $A=A_1\otimes A_2$ with $\exp(A_2)=p$ and $\mathrm{ind}(A_1)=\exp(A_1)=p^b$. Then $\mathrm{CH}^2(X)$ is torsion free.*

Proof. We prove this claim by induction on the index of the factor A_1 over any field extension. Our induction hypothesis will be that the claim holds for all algebras B, over any field extension of k, that are Brauer equivalent to a division algebra D_B that decomposes into a product $D_B=D_1\otimes D_2$ with $\exp(D_2)=p$ and $\mathrm{ind}(D_1)=p^a$ for any $a<b$. The starting point for this induction is the case that $a=1$; in other words, we need to show that for any algebra B Brauer equivalent to a division algebra D_B admitting a decomposition $D_1\otimes D_2$ satisfying $\exp(D_2)=p$, and $\mathrm{ind}(D_1)=\exp(D_1)=p$, the Chow group $\mathrm{CH}^2(\mathrm{SB}(B))$ is torsion free. This has already been done; it suffices to check $\mathrm{CH}^2(\mathrm{SB}(D_B))$ is torsion free and this is proved in [Kar98, Proposition 5.3].

To check whether $\mathrm{CH}^2(X)$ is torsion free, it suffices to assume that k is p-special. In this case, there is a field F/k with $[F:k]=p$ and $\mathrm{ind}(A_{1,F})=p^{b-1}$. Because the exponent $\exp(A_1)$ divides $[L:k]\exp(A_{1,L})$ for any finite field extension L/k by [FD93, Chapter 4, Ex. 13 (c)] we also have that $\exp(A_{1,F})=p^{b-1}$; this will allow us to use our induction hypothesis.

We note that

$$r\mathcal{B}eh(A) = (n, b-1, b-2, \ldots, 1, 0) \quad \text{and} \quad r\mathcal{B}eh(A_F) = (n-1, b-2, \ldots, 0)$$

so $S_{X_F} = \{1\} = S_X$ and $b - 1 \geq 1$. Thus, Lemma 4.2 below can be applied and the pushforward $\pi_{F/k,*} : \mathrm{Q}^2(X_F) \to \mathrm{Q}^2(X)$, that exists by Proposition 3.6, is a surjection. By the projection formula one can also determine that, since $\pi^*_{F/k} : \mathrm{CT}^2(1; X) \to \mathrm{CT}^2(1; X_F)$ has image in $p\mathrm{CT}^2(1; X_F)$, the left vertical arrow in the diagram below is surjective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CT}^2(1; X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{CH}^2(X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{Q}^2(X_F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{F/k,*}} & & \downarrow{\scriptstyle \pi_{F/k,*}} & & \downarrow{\scriptstyle \pi_{F/k,*}} & & \\
0 & \longrightarrow & \mathrm{CT}^2(1; X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{CH}^2(X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{Q}^2(X) & \longrightarrow & 0
\end{array}$$

The claim follows from the snake lemma since, by induction, the group $\mathrm{CH}^2(X_F) \otimes \mathbb{Z}_{(p)}$ is torsion free. □

Lemma 4.2. *Assume p is an odd prime. Let A be a central simple algebra. Assume F/k is a finite extension of degree $[F : k] = p^m$, with $m \geq 1$. Write $n_i = v_p\mathrm{rk}(\zeta_X(p^i))$ and $n'_i = v_p\mathrm{rk}(\zeta_{X_F}(p^i))$. Then, if one has an inequality*

$$n'_j + m \leq n_j$$

for all $j \in S_X$, the pushforward $\pi_{F/k,*} : \mathrm{Q}^2(X_F) \to \mathrm{Q}^2(X)$ is surjective.

Proof. It's sufficient to show, in light of Proposition 3.7, that $\pi_{F/k,*}$ takes the elements $\tau^{X_F}_{p^i}(2)$ of $\mathrm{CT}^2(p^i; X_F) \otimes \mathbb{Z}_{(p)}$ to the elements $\tau^X_{p^i}(2)$ of $\mathrm{CT}^2(p^i; X) \otimes \mathbb{Z}_{(p)}$ for all $i \in S_X$. Since the composition $\pi_{F/k,*} \circ \pi^*_{F/k} = [F : k] = p^m$ is multiplication by p^m by Lemma 3.4, it is the same as proving that p^m divides the coefficient c in the expression

$$\text{(c)} \qquad \pi^*_{F/k}(\tau^X_{p^i}(2)) = c\tau^{X_F}_{p^i}(2)$$

for any $i \in S_X$.

To do this we just expand the expression by hand. Since for any $j \in S_X$, the term n_j can't equal 0 by assumption, all of the elements we consider are second Chern classes. Then

$$\begin{aligned}
\pi^*_{F/k}\left(c_t(\zeta_X(p^i))\right) &= c_t(\zeta_{X_F}(p^i))^{p^{n_i - n'_i}} \\
&= \left(1 + \tau^{X_F}_{p^i}(1)t + \cdots + \tau^{X_F}_{p^i}(p^{n_i})t^{p^{n_i}}\right)^{p^{n_i - n'_i}} \\
&= 1 + p^{n_i - n'_i}\tau^{X_F}_{p^i}(1)t + \left(p^{(n_i - n'_i)}\tau^{X_F}_{p^i}(2) + \binom{p^{n_i - n'_i}}{2}\tau^{X_F}_{p^i}(1)^2\right)t^2 + \cdots.
\end{aligned}$$

Since p^m divides $p^{n_i - n'_i}$, and since $\tau_{p^i}(1)^2$ is a multiple of $\tau_{p^i}(2)$, it follows that p^m divides the coefficient of c in the expression (c) as claimed. □

Example 4.3. Theorem 4.1 can be used to construct central simple algebras A of index n and exponent m with $\mathrm{CH}^2(\mathrm{SB}(A)) = \mathbb{Z}$ for any pair of odd integers (n, m) with m dividing n and having the same prime factors.

For example, let p^a be the highest power of p dividing n and let p^b be the highest power of p dividing m. One can find, over say $k = \mathbb{Q}$, a division algebra D_0 having $\mathrm{ind}(D_0) = \exp(D_0) = p^b$, and division algebras $D_1, \ldots, D_{a-b}$ (when $a \neq b$) each of degree p with $D^p = D_0 \otimes D_1 \otimes \cdots \otimes D_{a-b}$ satisfying $\mathrm{ind}(D^p) = p^a$. Then taking A to be the product of the D^p over all primes p dividing n gives an algebra with $\mathrm{CH}^2(\mathrm{SB}(A)) = \mathbb{Z}$.

Remark 4.4. The techniques above can also be used to (sometimes) get a geometric version of the statement: if A is Brauer equivalent to a decomposable division algebra over k, then A_F is Brauer equivalent to a decomposable division algebra over any extension F/k.

To do this, let $X = \mathrm{SB}(A)$ and let $\pi : X_F \to X$ be the projection. Assume that $\mathrm{Q}^2(X) = 0$, assume either S_{X_F} is empty or there is an equality of sets $S_X = S_{X_F}$ and, in the latter case, assume $\mathrm{rk}(\zeta_X(p^i)) = \mathrm{rk}(\zeta_{X_F}(p^i))$ for every $i \in S_X$. Then the pullback $\pi^*_{F/k} : \mathrm{Q}^2(X) \to \mathrm{Q}^2(X_F)$ is surjective; this doesn't require $\pi_{F/k,*}$ but only the description of $\mathrm{Q}^2(X)$ given in Proposition 3.7.

This applies, for example, when A is a division algebra of index p^2 and exponent p, for an odd prime p, that decomposes into the tensor product of two division algebras of index p.

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