THE CONIVEAU FILTRATION ON K_1 FOR SOME SEVERI-BRAUER VARIETIES

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ABSTRACT. We produce an isomorphism $E_\infty^{m,-m-1} \cong \mathrm{Nrd}_1(A^{\otimes m})$, between terms of the K-theory coniveau spectral sequence of a Severi-Brauer variety X associated to a central simple algebra A and a reduced norm group, assuming: A has equal index and exponent over all finite extensions of its center, and $\mathrm{SK}_1(A^{\otimes i}) = 1$ for all $i > 0$.

Notation and conventions.
We work over a fixed base field k.
A variety is a separated scheme of finite type over a field.
For a prime p, we write $v_p(-)$ for the p-adic valuation.

1. Introduction

Some K-cohomology groups were studied, and computed, for Severi-Brauer varieties associated to algebras with square-free degree in [MS82]. As an application of these computations one can compute the Chow groups of these Severi-Brauer varieties and find they are torsion free. Chow groups of arbitrary Severi-Brauer varieties X have been studied in depth and, in certain degrees, are known to be torsion free (e.g. $\mathrm{CH}^0(X)$ is free trivially, $\mathrm{CH}^1(X)$ is torsion free by [Art82], $\mathrm{CH}_0(X)$ is torsion free by [CM06], if X is associated to an algebra whose index equals its exponent then $\mathrm{CH}^2(X)$ is torsion free by [Kar98]).

The Chow groups of Severi-Brauer varieties are not always torsion free. Their torsion subgroups have also been studied in depth. In [Kar98], Karpenko shows, if X is a Severi-Brauer variety associated to an algebra with differing index and exponent, $\mathrm{CH}^2(X)$ sometimes contains a nontrivial torsion subgroup which surjects onto torsion in the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$. In a different direction, Merkurjev [Mer95] has shown that there is sometimes nontrivial torsion in the Chow groups of Severi-Brauer varieties which occurs in codimension 3 or higher; this torsion can't be detected by Karpenko's methods since it's contained in the kernel of the canonical epimorphism from $\mathrm{CH}(X)$ onto the graded group associated with the coniveau filtration on the Grothendieck group $\mathrm{G}_0(X)$.

Recently, Karpenko has computed the Chow ring of a Severi-Brauer variety associated to a central simple algebra with equal index and exponent under the assumption the Chow ring is generated by Chern classes, [Kar17]. In this computation, the Chow ring turns out to be torsion free. Without the assumption the Chow ring is generated by Chern classes, any nontrivial torsion in the Chow ring of such a Severi-Brauer variety will come from nontrivial differentials in the K-theory coniveau, or Brown-Gersten-Quillen, spectral sequence.

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This article stemmed from exploring the possibility of torsion in the Chow group of a Severi-Brauer variety associated to an algebra A with index equal to its exponent. Hopefully, it will be of use in further study of this problem.

Section 2 is mainly for reference and introducing notation. In Section 3 we prove a series of lemmas that will be used for the main results of Sections 4 and 5.

In Section 4, we compute the $E_\infty^{m,-m-1}$ terms of the K-theory coniveau spectral sequence for any Severi-Brauer variety X associated to an algebra A satisfying the properties: the index of A is a power of a prime p, the exponent of A equals the index of A over all finite extensions of the center of A, and the reduced Whitehead groups $\mathrm{SK}_1(A^{\otimes r}) = 1$ vanish for all $r \geq 1$. This result is a direct generalization of the known computation for the terms $E_\infty^{m,-m}$ and the proof of the main theorem manages to describe both simultaneously. The main result is Theorem 4.2; it's proof is elementary but, it requires some involved arguments comparing the reduced norms of certain tensor powers of a given algebra.

In Section 5, we show how to prove the general case stated in the abstract using the primary case of Section 4.

2. On the K-theory of a Severi-Brauer variety

The material in this section has been developed in detail by Quillen, [Qui73]. The K-theory coniveau spectral sequence, or the Brown-Gersten-Quillen spectral sequence, is a fourth quadrant cohomological spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} \mathrm{K}_{-p-q}(k(x)) \implies \mathrm{G}_{-p-q}(X)$$

where $X^{(p)}$ denotes the set of codimension p points of X. For a variety X, the spectral sequence converges, and for a regular variety X one can identify the E_2-terms with K-cohomology groups

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{K}_{-q}) \implies \mathrm{G}_{-p-q}(X).$$

Recall the K-cohomology groups $\mathrm{H}^p(X, \mathcal{K}_q)$ are defined to be the homology of a complex

$$\mathrm{H}^p(X, \mathcal{K}_q) = \mathrm{H}\left(\coprod_{x \in X^{(p-1)}} \mathrm{K}_{q-p+1}(k(x)) \to \coprod_{x \in X^{(p)}} \mathrm{K}_{q-p}(k(x)) \to \coprod_{x \in X^{(p+1)}} \mathrm{K}_{q-p-1}(k(x)) \right).$$

In particular, the groups $\mathrm{H}^p(X, \mathcal{K}_q) = 0$ whenever $p > q$ or $p > \dim(X)$.

The coniveau filtration is the filtration appearing in the abutment of the K-theory coniveau spectral sequence. If X is a regular variety (which is all that is worked with in this note), then there are natural isomorphisms $\mathrm{K}_i(X) \cong \mathrm{G}_i(X)$ and by transporting the filtration on G-theory to K-theory we get a coniveau filtration on the groups $\mathrm{K}_i(X)$. The jth term of this filtration on $\mathrm{K}_i(X)$ is denoted $\mathrm{K}_i(X)^j$ below. We write $\mathrm{K}_i(X)^{j/j+1}$ for the quotient $\mathrm{K}_i(X)^j/\mathrm{K}_i(X)^{j+1}$.

The K-theory of a Severi-Brauer variety X associated to a central simple algbera A was computed by Quillen in terms of the tautological bundle ζ_X on X:

Theorem 2.1 ([Qui73, §8, Theorem 4.1]). *Let X be the Severi-Brauer variety of a central simple algebra A. Then, for every $i \geq 0$ the group homomorphism*

$$\bigoplus_{j=0}^{\deg(A)-1} \mathrm{K}_i(A^{\otimes j}) \to \mathrm{K}_i(X)$$

induced by the exact functor that takes a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

Crucial in our computation will be the reduced norm subgroups of a central simple k-algebra. For this, let L be a Galois splitting field for A. The *first reduced norm* of A is defined to be the unique map making the following diagram commutative.

$$\begin{array}{ccc} \mathrm{K}_1(A_L) & \xrightarrow{\det} & \mathrm{K}_1(L) \\ \uparrow & & \uparrow \\ \mathrm{K}_1(A) & \xrightarrow{\mathrm{Nrd}_1} & \mathrm{K}_1(k) \end{array}$$

The vertical arrows in this diagram are induced by extension of scalars. Similarly we define the *zeroth reduced norm* of A to be the map $\mathrm{Nrd}_0 : \mathrm{K}_0(A) \to \mathrm{K}_0(k)$ taking the class of an A-module M to the k-vector space of dimension $\mathrm{rdim}_A(M)$, the reduced dimension of M. For $i = 0, 1$ we will often use the abbreviation $\mathrm{Nrd}_i(\mathrm{K}_i(A)) := \mathrm{Nrd}_i(A)$.

The kernel of the map Nrd_i is called the *ith reduced Whitehead group* and denoted $\mathrm{SK}_i(-)$. Note the group $\mathrm{SK}_0(A)$ necessarily vanishes since Nrd_0 is injective with image the subgroup generated by the index of A, $\mathrm{ind}(A)\mathbb{Z} \subset \mathrm{K}_0(k) = \mathbb{Z}$. The group $\mathrm{SK}_1(A)$ doesn't vanish in general.

For any finite field extension E of k, the extension of scalars map $\rho^*_{E/k} : \mathrm{K}_i(X) \to \mathrm{K}_i(X_E)$ is the sum of the maps $\mathrm{K}_i(A^{\otimes j}) \to \mathrm{K}_i(A_E^{\otimes j})$ in the decomposition of Theorem 2.1. In the other direction, the pushforward $\rho_{E/k*} : \mathrm{K}_i(X_E) \to \mathrm{K}_i(X)$ is given by the sum of the norm maps $\mathrm{K}_i(A_E^{\otimes j}) \to \mathrm{K}_i(A^{\otimes j})$ in the same decomposition. If $i = 0$ then the norm map is characterized componentwise by having image the number

$$\rho_{E/k*}(\mathrm{K}_0(A_E)) = [E : k]\frac{\mathrm{rdim}_{A_E}(M)}{\mathrm{rdim}_A(N)} \subset \mathrm{K}_0(A) = \mathbb{Z}$$

where M, N are simple modules under A_E, A respectively. The image of the norm maps when $i = 1$ are more complicated to describe. In the simple situation we work in, these images can be described fairly explicitly. We do this in detail in the next section.

3. Relations between reduced norms

In this section we fix a central simple algebra A over k and we set X to be the Severi-Brauer variety associated with X.

Our first objective is to describe the image of the reduced norm using splitting fields of A:

Lemma 3.1. *Let A be a central simple algebra. Then, for every finite field extension L of k and for $i = 0, 1$, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{K}_i(A_L) & \xrightarrow{\mathrm{Nrd}_i} & \mathrm{K}_i(L) \\ \downarrow{\scriptstyle N_{A_L/A}} & & \downarrow{\scriptstyle N_{L/k}} \\ \mathrm{K}_i(A) & \xrightarrow{\mathrm{Nrd}_i} & \mathrm{K}_i(k) \end{array}$$

where both $N_{A_L/A}$ and $N_{L/k}$ are the norm maps induced by restriction of scalars.

Moreover, the subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extensions of k that split A. This can be reduced further: the subgroup $\mathrm{Nrd}_i(A)$ is generated by the images $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extensions of k that are maximal subfields of the underlying division algebra of A.

Proof. The commutativity of the digram is clear when $i = 0$, and is well-known (see [GS06, Proposition 2.8.11]) when $i = 1$.

The only claim that needs to be proved is the last one: the subgroup $\mathrm{Nrd}_i(A)$ is generated by norms of maximal subfields of the underlying division algebra of A. In the case $i = 0$, the claim follows from the fact such a field has degree $\mathrm{ind}(A)$ over k so we are left proving the case $i = 1$.

For the proof when $i = 1$, we'll use Morita invariance to reduce to the case A is a division algebra and we'll use [GS06, Proposition 2.6.3] which says $\mathrm{Nrd}_1(x) = N_{K/k}(x)$ for any element x of a maximal subfield K contained in A. Any element x of A is contained in some maximal subfield (indeed, if F is a maximal element in the collection of subfields of A containing $k(x)$, then the centralizer of F in A is F itself – this is known to be equivalent to being a maximal subfield) so taking the composition

$$A^\times \twoheadrightarrow \mathrm{K}_1(A) \xrightarrow{\mathrm{Nrd}_1} \mathrm{K}_1(k)$$

of the natural surjection and the reduced norm gives the result by the commutativity of the given diagram. □

The K-theory of the Severi-Brauer variety X relies heavily on the tensor powers of the algebra A due to the decomposition of Theorem 2.1. Because of this, we'll need to investigate certain relations between the reduced norms $\mathrm{Nrd}_i(A)$ and $\mathrm{Nrd}_i(A^{\otimes r})$ for varying $r \geq 0$. It will be necessary in our formulation of these relations to introduce some condition on the index of A over finite extensions. From now on we'll say an algebra A satisfies condition (C) if:

$$\text{(C)} \qquad \mathrm{ind}(A_E) = \exp(A_E) \text{ for any finite extension } E/k.$$

Example 3.2. Any central simple algebra of square-free index satisfies condition (C) trivially. Any central simple algebra over a finite extension of $\mathbb{Q}_p$ satisfies condition (C). Central simple algebras over function fields of surfaces, with base a separably closed field, having index coprime to the characteristic of the base also satisfy condition (C), see [dJ04].

Moreover, if a central simple algebra A satisfies condition (C) then so do the tensor powers of A. This is because, given a central simple algebra A with equal index and exponent, the indices of all tensor powers of A can be explicitly determined. If the index of A was a power of a prime p, say p^n, then $A^{\otimes p}$ has index p^{n-1}, cf. [Kar98, Example 3.9]. The general case follows easily from this one.

Remark 3.3. There exists a cyclic algebra A of index and exponent 4, over a field F of characteristic 2, along with a finite purely inseparable field extension E/F with $[E : F] = 2$ and such that $\operatorname{ind}(A_E) = 4$ and $\exp(A_E) = 2$ (cf. [Per41, Theorem 4]).

Lemma 3.4. *Let A be a central simple k-algebra with $\operatorname{ind}(A) = p^n$ for some $n \geq 0$ and let $i = 0$ or $i = 1$. Then*

$$\operatorname{Nrd}_i(A^{\otimes j}) = \operatorname{Nrd}_i(A^{\otimes p^{v_p(j)}})$$

for any $j > 0$.

Proof. By Lemma 3.1 the subgroup $\operatorname{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extension of k splitting $A^{\otimes j}$. The set of such fields is the same for $A^{\otimes j}$ and $A^{\otimes p^{v_p(j)}}$, which proves the claim. □

Lemma 3.5. *Let A be a central simple k-algebra with $\operatorname{ind}(A) = p^n = \exp(A)$ for some prime p and some $n \geq 0$. Assume A satisfies condition* (C)*. Then for $i = 0, 1$ the containments*

$$\operatorname{Nrd}_i(A^{\otimes p^a}) \supset \operatorname{Nrd}_i(A^{\otimes p^b}) \supset \operatorname{Nrd}_i(A^{\otimes p^a})^{p^{a-b}}$$

hold for all $a \geq b \geq 0$.

Proof. By Lemma 3.1 the subgroup $\operatorname{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(k)$ is generated by the norm subgroups $N_{L/k}(\mathrm{K}_i(L))$ as L varies over all finite extension of k splitting $A^{\otimes j}$. If such an L would split $A^{\otimes p^b}$, then L would also split $A^{\otimes p^a}$. Hence we have the inclusion $\operatorname{Nrd}_i(A^{\otimes p^b}) \subset \operatorname{Nrd}_i(A^{\otimes p^a})$.

To show the inclusion $\operatorname{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset \operatorname{Nrd}_i(A^{\otimes p^b})$, we work in two cases. If $a \geq n$, then $A^{\otimes p^a}$ is split; if L is a maximal subfield of the underlying division algebra of $A^{\otimes p^b}$, then $[L : k] = p^{n-b}$ (see Example 3.2) and

$$\operatorname{Nrd}_i(A^{\otimes p^a})^{p^{a-b}} \subset p^{n-b}\mathrm{K}_i(k) = N_{L/k}(\mathrm{K}_i(k)) \subset \operatorname{Nrd}_i(A^{\otimes p^b}).$$

Otherwise, when $a < n$, let L be a maximal subfield of the underlying division algebra of $A^{\otimes p^a}$. Then L has degree $[L : k] = p^{n-a}$, the algebra A_L has exponent dividing p^a and, since we're assuming condition (C), index dividing p^a. If E is a maximal subfield of the underlying division algebra of $A_L^{\otimes p^b}$ then $[E : L]$ divides p^{a-b}. Again by Lemma 3.1 we have the inclusion

$$N_{E/k}(\mathrm{K}_i(E)) \subset \operatorname{Nrd}_i(A^{\otimes p^b})$$

since E splits $A^{\otimes p^b}$. It follows that for any element x of $\mathrm{K}_i(L) \subset \mathrm{K}_i(E)$ we have

$$N_{E/k}(x) = N_{L/k}(N_{E/L}(x)) = N_{L/k}(x^{[E:L]}) = N_{L/k}(x)^{[E:L]}$$

is contained in $\operatorname{Nrd}_i(A^{\otimes p^b})$. The proof is then complete since we've shown the collection of elements $N_{L/k}(x)^{p^{a-b}}$, as L varies over all maximal subfields of the underlying division algebra of $A^{\otimes p^a}$ and x varies over $\mathrm{K}_i(L)$, are contained in $\operatorname{Nrd}_i(A^{\otimes p^b})$ and these form a generating set by Lemma 3.1. □

Lemma 3.6. *Let A be a central simple k-algebra with $\operatorname{ind}(A) = p^n = \exp(A)$ for some prime p and some $n \geq 0$. Assume A satisfies condition* (C)*. Then for $i = 0, 1$ there is containment*

$$\operatorname{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \operatorname{Nrd}_i(A^{\otimes b})$$

for all $a \geq b > 0$.

Proof. The proof continues by working in cases: assuming either $v_p(a) \leq v_p(b)$ or $v_p(a) > v_p(b)$. In the first case, $v_p(a) \leq v_p(b)$, we appeal to Lemma 3.4 and Lemma 3.5 to find

$$\mathrm{Nrd}_i(A^{\otimes a}) = \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}}) \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b}).$$

In the second case, $v_p(a) > v_p(b)$, we appeal to the second containment of Lemma 3.5. That is to say, by Lemma 3.7 below we find $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$ so that

$$\mathrm{Nrd}_i(A^{\otimes a})^{\binom{a}{b}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(a)}})^{p^{v_p(a)-v_p(b)}} \subset \mathrm{Nrd}_i(A^{\otimes p^{v_p(b)}}) = \mathrm{Nrd}_i(A^{\otimes b})$$

by applying Lemma 3.4 for the first inclusion, Lemma 3.5 for the second inclusion, and Lemma 3.4 for the last equality. □

The lemma needed for the above is:

Lemma 3.7. *Assume $a > b$ and $v_p(a) > v_p(b)$. Then $v_p(\binom{a}{b}) \geq v_p(a) - v_p(b)$.*

Proof. More generally, for any pair of integers $a > b$, one can show $\frac{a}{(a,b)}$ divides the binomial coefficient $\binom{a}{b}$. The claim follows from noting

$$v_p\left(\frac{a}{(a,b)}\right) = v_p(a) - v_p((a,b)) = v_p(a) - v_p(b).$$

First, write $(a,b) = na + mb$ with n, m both integers. Then

$$\frac{(a,b)}{a}\binom{a}{b} = \frac{(na+mb)}{a}\binom{a}{b} = n\binom{a}{b} + \frac{mb}{a}\binom{a}{b} = n\binom{a}{b} + m\binom{a-1}{b-1}$$

with the latter sum an integer. □

To go from an algebra of p-primary index to an arbitrary central simple algebra A, see Proposition 5.1, we'll need a characterization of $\mathrm{Nrd}_i(A)$ in terms of the primary components of A. For this, we fix a primary decomposition

$$A \cong A_{p_1} \otimes \cdots \otimes A_{p_s}$$

with $p_1, ..., p_s$ the primes dividing $\mathrm{ind}(A)$ (such decompositions exist with the factors unique up to isomorphism, see [GS06, Proposition 4.5.16]). For each algebra A_{p_j} we fix a maximal subfield F_{p_j} of its underlying division algebra, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, ..., F_{p_{j-1}}, F_{p_{j+1}}, ..., F_{p_s}$, the jth field being omitted, contained in some fixed algebraic closure L.

Lemma 3.8. *In the notation above, and for $i = 0, 1$,*

$$\mathrm{Nrd}_i(A) = \bigcap_{j=1}^{s} \mathrm{Nrd}_i(A_{F^{p_j}})$$

inside of $\mathrm{K}_i(L)$.

Proof. If $s = 1$, the lemma is trivial so we can assume $s > 1$.

The inclusion $\subset$ is immediate from Lemma 3.1 since a field E splitting A also necessarily splits each of the $A_{F^{p_j}}$.

For the other inclusion, $\supset$, we let x be an element of the intersection. By Lemma 3.1 this means we have equalities

$$x = N_{E_{1,1}/F^{p_1}}(y_{1,1}) \cdots N_{E_{1,r_1}/F^{p_1}}(y_{1,r_1})$$

$$\vdots$$

$$x = N_{E_{s,1}/F^{p_s}}(y_{s,1}) \cdots N_{E_{s,r_s}/F^{p_s}}(y_{s,r_s})$$

for some elements $y_{j,k}$ of fields $E_{j,k}$ splitting $A_{F^{p_j}}$ respectively. It follows from these equalities that x is an element of $B = \mathrm{K}_i(F^{p_1}) \cap \cdots \cap \mathrm{K}_i(F^{p_s})$. If $i = 0$, then B is just $\mathrm{ind}(A)\mathbb{Z}$. If $i = 1$, then, since by construction the degrees $[F^{p_j} : k]$ are divisible by all primes dividing $\mathrm{ind}(A)$ except for p_j, we have $\gcd([F^{p_1} : k], ..., [F^{p_s} : k]) = 1$ and $B = k^{\times}$.

Applying the norm, from F^{p_j} to k, to the corresponding expression above for x, we find the elements

$$N_{F^{p_j}/k}(x) = N_{E_{j,1}/k}(y_{j,1}) \cdots N_{E_{j,r_j}/k}(y_{j,r_j})$$

are contained in $\mathrm{Nrd}_i(A)$, for every $1 \le j \le s$, since each $E_{j,k}$ splits $A_{F^{p_j}}$ and so necessarily also splits A. Since x is already contained in $\mathrm{K}_i(k)$, taking the norm also yields equalities

$$N_{F^{p_j}/k}(x) = x^{[F^{p_j}:k]}.$$

Finally, as x is in the subgroup spanned by these powers, x is contained in $\mathrm{Nrd}_i(A)$, completing the proof. □

4. The coniveau filtration on K_i for a p-primary algebra

We fix a prime p throughout. We fix a central simple algebra A with index $\mathrm{ind}(A) = p^n$ and exponent $\exp(A) = p^n$ for some $n > 0$. We write X for the Severi-Brauer variety of A.

This section describes the groups $\mathrm{K}_i(X)^j$ and $\mathrm{K}_i(X)^{j/j+1}$ for $j \ge 0$ assuming A satisfies condition (C) and either $i = 0$ or, $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \ge 1$. In the case $i = 0$, this result was shown in [Kar98, Proposition 3.3] (condition (C) is not needed in this result). Although the only new result is when $i = 1$, the proof does not depend on this assumption.

We note that the assumption $\mathrm{SK}_1(A^{\otimes r})$ is trivial for all powers r is another way of stating that $\mathrm{K}_1(X) \to \mathrm{K}_1(X_L)$ is injective for a splitting field L of A. The reason the latter, more natural, assumption is not given is because it's often easier to check that the groups $\mathrm{SK}_1(A^{\otimes r})$ are trivial. Note the analogous statement is also true replacing $i = 1$ with $i = 0$ in the above so that the map $\mathrm{K}_0(X) \to \mathrm{K}_0(X_L)$ is always injective. Formally:

Lemma 4.1. *Suppose B is an arbitary central simple algebra and let Y be the Severi-Brauer variety of B. Let L be a splitting field for B. Then, for $i = 0, 1$ the pullback $\mathrm{K}_i(Y) \to \mathrm{K}_i(Y_L)$ is injective if, and only if, the groups $\mathrm{SK}_i(B^{\otimes j})$ are trivial for all $j \ge 0$.*

Proof. The diagram

$$\begin{array}{ccc} \mathrm{K}_i(B_L^{\otimes r}) & \xrightarrow[\quad]{\mathrm{Nrd}_i} & \mathrm{K}_i(L) \\ \pi_r^* \uparrow & & \uparrow \\ \mathrm{K}_i(B^{\otimes r}) & \xrightarrow{\mathrm{Nrd}_i} & \mathrm{K}_i(k) \end{array}$$

commutes where the vertical arrows are the extension of scalars maps. Since the right-vertical arrow is always an injection we find $\mathrm{SK}_i(B^{\otimes r}) = \ker(\pi_r^*)$. The claim then follows from Theorem 2.1 by summing over all $r \ge 0$. □

As in the above lemma, let B be an arbitrary central simple algebra and Y the associated Severi-Brauer variety. If L is a splitting field for B, then $\mathrm{K}_0(Y_L)$ is generated as a group by the powers γ^i, from $i = 0$ to $\deg(B) - 1$, of the element γ representing the class of $\mathcal{O}_{Y_L}(-1)$.

By Lemma 4.1, the pullback $\mathrm{K}_0(Y) \to \mathrm{K}_0(Y_L)$ is injective and we identify $\mathrm{K}_0(Y)$ with its image in $\mathrm{K}_0(Y_L)$. Similarly, the group $\mathrm{K}_1(Y_L)$ is a sum of groups $L^\times\gamma^i$ as i ranges from $i = 0$ to $i = \deg(B) - 1$. If $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 1$, then the pullback $\mathrm{K}_1(Y) \to \mathrm{K}_1(Y_L)$ is injective and we identify $\mathrm{K}_1(Y)$ with its image in $\mathrm{K}_1(Y_L)$.

Theorem 4.2. *Assume A satisfies condition* (C)*. Let L be a splitting field for A. If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(A^{\otimes r}) = 1$ for all $r \geq 1$, then there is an equality (with notation as above)*

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}$$

for all $0 \leq j \leq \deg(A) - 1$. For $j < 0$, or for $j > \deg(A) - 1$, the groups $\mathrm{K}_i(X)^j = 0$ vanish.

Proof. The claim when $j < 0$ or $j > \deg(A) - 1$ is immediate: the first of these is by definition, the second follows from the fact $(\gamma - 1)^{\deg(A)} = 0$ in $\mathrm{K}_0(X)$. Recall (cf. [Pey95, Proposition 3.6]) the coniveau filtration on $\mathrm{K}_i(X_L)$ is given by

$$\mathrm{K}_i(X_L)^j = \mathrm{K}_i(A_L^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{K}_i(A_L^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}$$

where $\gamma = [\mathcal{O}(-1)]$ is the class of the tautological line bundle in $\mathrm{K}_0(X_L)$. Under the pullback $\mathrm{K}_i(X) \to \mathrm{K}_i(X_L)$ the groups $\mathrm{K}_i(A^{\otimes j})$ are identified with the subgroups $\mathrm{Nrd}_i(A^{\otimes j}) \subset \mathrm{K}_i(L)$. Hence, we identify

$$\mathrm{K}_i(X) = \mathrm{Nrd}_i(k) \cdot 1 + \mathrm{Nrd}_i(A)\gamma + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})\gamma^{\deg(A)-1}.$$

We claim

$$(*) \qquad \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma - 1)^{\deg(A)-1}.$$

The proof utilizes the following lemmas:

Lemma 4.3. *Let A and L be as in Theorem 4.2. Fix an elemenet b in $\mathrm{Nrd}_i(A^{\otimes k})$ with $k \geq 0$ and $i = 0$ or $i = 1$. Then, for any sequence of integers $(n_j)_{j\geq 0}$ an equality*

$$bx^k = \sum_{j\geq 0} a_j(x + n_j)^j$$

inside of the free $\mathrm{K}_i(L)$-module $\mathrm{K}_i(L)[x]$ implies a_j is contained in $\mathrm{Nrd}_i(A^{\otimes j})$ for all $j \geq 0$.

Proof. By assumption $a_k = b$ is contained in $\mathrm{Nrd}_i(A^{\otimes k})$. By descending induction on j, we assume each a_j is contained in $\mathrm{Nrd}_i(A^{\otimes j})$ for all j larger than some fixed $l \geq 0$. Then by expanding the right side of the given equality and comparing coefficients yields

$$a_l = -\sum_{j=l+1}^{k} n_j^{j-l}\binom{j}{l}a_j$$

which is contained in $\mathrm{Nrd}_i(A^{\otimes l})$ due to Lemma 3.6 applied to each $\binom{j}{l}a_j$. □

Lemma 4.4. *Keeping notation as above, we have*

$$\sum_{j\geq 0} \mathrm{Nrd}_i(A^{\otimes j})\gamma^j = \sum_{j\geq 0} \mathrm{Nrd}_i(A^{\otimes j})(\gamma - 1)^j$$

inside of $\mathrm{K}_i(X_L)$.

Proof. Setting $n_j = -1$ for all $j \geq 0$ in Lemma 4.3, and setting $x = \gamma$, shows the forward containment. Setting $n_j = 1$ for all $j \geq 0$, and setting $x = \gamma - 1$, shows the reverse containment. □

Continuing with the proof of Theorem 4.2, we have

$$\begin{aligned}\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j &= \sum_{n\geq 0} \mathrm{Nrd}_i(A^{\otimes n})\gamma^n \cap \sum_{n\geq j} \mathrm{K}_i(L)(\gamma-1)^j \\ &= \sum_{n\geq 0} \mathrm{Nrd}_i(A^{\otimes n})(\gamma-1)^n \cap \sum_{n\geq j} \mathrm{K}_i(L)(\gamma-1)^n \\ &= \sum_{n\geq j} \mathrm{Nrd}_i(A^{\otimes n})(\gamma-1)^n\end{aligned}$$

as claimed. Here we used Lemma 4.4 to go from the first line to the second. □

Corollary 4.5. *Let L be an algebraic closure of k. Assume A satisfies condition* (C). *Let $i = 0$ or $i = 1$ and assume $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 1$. Then we have an equality*

$$\mathrm{K}_i(X)^j = \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$$

for all $j \geq 0$.

Proof. It's clear we have the inclusion $\mathrm{K}_i(X)^j \subset \mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$. By Theorem 4.2, there is an equality

$$\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j = \mathrm{Nrd}_i(A^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}.$$

To show the reverse containment $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X)^j$ we go by induction on the index. That is to say: if E is a finite extension of k splitting A then we have containment $\mathrm{K}_i(X_E) \cap \mathrm{K}_i(X_L)^j \subset \mathrm{K}_i(X_E)^j$ and for our induction hypothesis we assume this containment holds for all fields E with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$.

If E is a finite extension of k with $\mathrm{ind}(A_E) < \mathrm{ind}(A)$ then, using our induction hypothesis and the assumption A satisfies condition (C), we have

$$\begin{aligned}\mathrm{K}_i(X)^j &= \rho^*_{L/k}(\mathrm{K}_i(X)^j) \\ &\supset \rho^*_{L/k}(\rho_{E/k*}(\mathrm{K}_i(X_E)^j)) \\ &= \rho_{E/k_*}\left(\mathrm{Nrd}_i(A_E^{\otimes j})(\gamma-1)^j + \cdots + \mathrm{Nrd}_i(A_E^{\otimes \deg(A)-1})(\gamma-1)^{\deg(A)-1}\right).\end{aligned}$$

Expanding a product $(\gamma-1)^r$ and taking $\rho_{E/k*}$ shows

$$\rho_{E/k*}(a(\gamma-1)^r) = N_{E/k}(a)(\gamma-1)^r.$$

Since all elements of $\mathrm{Nrd}_i(A^{\otimes r})$ are norms from finite extensions E of k splitting $A^{\otimes r}$ by Lemma 3.1, it follows $\mathrm{K}_i(X) \cap \mathrm{K}_i(X_L)^j$ is generated by the groups on the right of the containment above. □

Corollary 4.6. *Let $i = 0$, or $i = 1$ and $\mathrm{SK}_i(A^{\otimes r}) = 1$ for all $r \geq 0$. Assume A satisfies condition* (C). *Then there is an isomorphism*

$$\mathrm{K}_i(X)^{j/j+1} \cong \mathrm{Nrd}_i(A^{\otimes j})$$

for all $0 \leq j \leq \deg(A) - 1$. For other j these groups vanish.

Proof. This follows immediately from Theorem 4.2 and Corollary 4.5. □

5. THE CONIVEAU FILTRATION ON K_i FOR A CENTRAL SIMPLE ALGEBRA

In this section we assume B is a central simple algebra with $\mathrm{ind}(B_E) = \exp(B_E)$ for all finite field extensions E/k. We let Y be the Severi-Brauer variety of B.

Proposition 5.1. *If $i = 0$, or if $i = 1$ and $\mathrm{SK}_1(B^{\otimes r}) = 1$ for all $r \geq 0$, then there is an isomorphism*

$$\mathrm{K}_i(Y)^{j/j+1} \cong \mathrm{Nrd}_i(B^{\otimes j})$$

for all $0 \leq j \leq \deg(B) - 1$. For other j these groups vanish.

Proof. Using a result of Karpenko, [Kar00, Example 10.20], we can assume B is a division algebra throughout the proof.

Fix a primary decomposition

$$B \cong B_{p_1} \otimes \cdots \otimes B_{p_s}$$

with $p_1, ..., p_s$ the primes dividing $\mathrm{ind}(B)$. We can assume $s > 1$, as the result has been proved above otherwise. For each algebra B_{p_j} we fix a maximal subfield F_{p_j} of its underlying division algebra, necessarily of degree a power of p_j over k. We set F^{p_j} to be a composite of the fields $F_{p_1}, ..., F_{p_{j-1}}, F_{p_{j+1}}, ..., F_{p_s}$, the jth field being omitted, contained in some fixed algebraic closure L of k

We first observe an equality

$$\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j = \mathrm{Nrd}_i(B^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}.$$

Indeed, by Lemma 3.8 and the explicit description of $\mathrm{K}_i(Y)$ given by Lemma 4.1, we have

$$\mathrm{K}_i(Y) = \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}})$$

inside of $\mathrm{K}_i(Y_L)$. Hence we get equalities

$$\begin{aligned}
\mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j &= \mathrm{K}_i(Y_{F^{p_1}}) \cap \cdots \cap \mathrm{K}_i(Y_{F^{p_s}}) \cap \mathrm{K}_i(Y_L)^j \\
&= \bigcap_{r=1}^{s} \left(\mathrm{K}_i(Y_{F^{p_r}}) \cap \mathrm{K}_i(Y_L)^j\right) \\
&= \bigcap_{r=1}^{s} \left(\mathrm{Nrd}_i(B_{F^{p_r}})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B_{F^{p_r}}^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}\right) \\
&= \mathrm{Nrd}_i(B^{\otimes j})(\gamma - 1)^j + \cdots + \mathrm{Nrd}_i(B^{\otimes \deg(B)-1})(\gamma - 1)^{\deg(B)-1}.
\end{aligned}$$

A careful reading of the proof of Corollary 4.5 shows that the assumption A has p-primary index was unnecessary. Hence the corollary can be applied to B as well to show $\mathrm{K}_i(Y) = \mathrm{K}_i(Y) \cap \mathrm{K}_i(Y_L)^j$ and the result follows. □

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