ON THE K-THEORY CONIVEAU EPIMORPHISM FOR PRODUCTS OF SEVERI-BRAUER VARIETIES

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Abstract. For X a product of Severi-Brauer varieties, we conjecture: if the Chow ring of X is generated by Chern classes, then the canonical epimorphism from the Chow ring of X to the graded ring associated to the coniveau filtration of the Grothendieck ring of X is an isomorphism. We show this conjecture is equivalent to: if G is a split semisimple algebraic group of type AC, B is a Borel subgroup of G and E is a standard generic G-torsor, then the canonical epimorphism from the Chow ring of E/B to the graded ring associated with the coniveau filtration of the Grothendieck ring of E/B is an isomorphism. In certain cases we verify this conjecture.

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise. Sometimes we use k as an index when no confusion will occur.

For any field F, we fix an algebraic closure $\overline{F}$.

A variety X is a separated scheme of finite type over a field.

Let $X = X_1 \times \cdots \times X_r$ be a product of varieties with projections $\pi_i : X \to X_i$. Let $\mathcal{F}_1, ..., \mathcal{F}_r$ be sheaves of modules on $X_1, ..., X_r$. We use $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_r$ for the external product $\pi_1^*\mathcal{F}_1 \otimes \cdots \otimes \pi_r^*\mathcal{F}_r$.

For a ring R with a $\mathbb{Z}$-indexed descending filtration $F_\nu^\bullet$, (e.g. $\nu = \gamma$ or τ as in Section 2), we write $\mathrm{gr}_\nu^i R$ for the corresponding quotient F_ν^i/F_ν^{i+1}. We write $\mathrm{gr}_\nu R = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}_\nu^i R$ for the associated graded ring.

A semisimple algebraic group G is of type AC if its Dynkin diagram is a union of diagrams of type A and type C. Similarly a semisimple group G is of type AA if its Dynkin diagram is a union of diagrams of type A.

For an index set $\mathcal{I}$, two elements $i, j \in \mathcal{I}$, we write δ_{ij} for the function which is 0 when $i \neq j$ and 1 if $i = j$.

Given two r-tuples of integers, say I, J, we write $I < J$ if the ith component of I is less than the ith component of J for any $1 \leq i \leq r$.

1. Introduction

For any smooth variety X, the coniveau spectral sequence for algebraic K-theory induces a canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ from the Chow ring of X to the associated graded ring of the coniveau filtration on the Grothendieck ring of X (for notation related to Grothendieck rings see Section 2). The kernel of this epimorphism is torsion, as can be seen using the Grothendieck-Riemann-Roch without denominators. In general this can't be refined: there are examples of smooth varieties where the kernel of the K-theory coniveau epimorphism is nontrivial. With this in mind, a particularly difficult problem has been finding families of varieties where this epimorphism is, or fails to be, an isomorphism. In this direction we propose the following:

Conjecture 1.1. *Let X be a product of Severi-Brauer varieties. If the Chow ring* $\mathrm{CH}(X)$ *is generated by Chern classes, then the canonical epimorphism* $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

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Since the ring $\mathrm{gr}_\tau \mathrm{G}(X)$ is computable for such X (see Section 2 for recollections on the Grothendieck rings of Severi-Brauer varieties and their products), a positive answer to Conjecture 1.1 could then be interepreted as a method for computing the Chow ring of such varieties. This is carried out, for instance, in [Kar17a, Theorem 3.1] where the first named author shows a special case of Conjecture 1.1 and, using this, is able to compute the Chow ring of certain generic Severi-Brauer varieties.

In Section 3, we give some evidence that a positive answer to Conjecture 1.1 is a likely one. The main result of this section, Theorem 3.3, shows that Conjecture 1.1 is equivalent to a particular case of an older conjecture of the first named author: [1]

Conjecture 1.2. *Let G be a split semisimple algebraic group, E a standard generic G-torsor, and P a special parabolic subgroup of G. Then the canonical epimorphism $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism.*

The proof uses an analysis of the products of Severi-Brauer varieties one obtains from a standard generic G-torsor for algebraic groups of type AA along with various specialization maps.

In Appendix A, we introduce the notion of the level of a central simple algebra. We show how the level gives a useful description of the Grothendieck ring of a Severi-Brauer variety and use this description in the main result of this section, Theorem A.15, where we prove Conjecture 1.1 for a single Severi-Brauer variety associated to a central simple algebra of level 1. This generalizes the previously known results obtained in [Kar17a, Theorem 3.1].

2. Grothendieck Rings of Severi-Brauer varieties

By $\mathrm{K}(X)$, we mean the Grothendieck ring of locally free sheaves (equivalently vector bundles) on a variety X; by $\mathrm{G}(X)$ we mean the Grothendieck group of coherent sheaves on X. The ith term of the γ-filtration on $\mathrm{K}(X)$ is denoted $F^i_\gamma(X)$; the ith term of the coniveau filtration on $\mathrm{G}(X)$ is denoted $F^i_\tau(X)$.

There's a canonical map $\varphi_X : \mathrm{K}(X) \to \mathrm{G}(X)$ taking the class $[\mathcal{L}] \in \mathrm{K}(X)$ of a locally free sheaf $\mathcal{L}$ to the class $[\mathcal{L}] \in \mathrm{G}(X)$. When X is smooth, φ_X is an isomorphism giving $\mathrm{G}(X)$ the structure of a ring. The coniveau filtration is compatible with the ring structure on $\mathrm{G}(X)$, and $\varphi_X(F^i_\gamma(X)) \subset F^i_\tau(X)$. Moreover, if the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes, then $\varphi_X(F^i_\gamma(X)) = F^i_\tau(X)$, cf. [Kar98, Proof of Theorem 3.7].

We will often be working with the rings $\mathrm{K}(X)$ for X a Severi-Brauer variety and for X a product of Severi-Brauer varieties.

In the case X is a Severi-Brauer variety, $\mathrm{K}(X)$ has been determined by Quillen. To state this result, recall that X is the variety of right ideals of dimension $\deg(A)$ in the central simple algebra A associated with X. The tautological vector bundle ζ_X on X is a right A-module.

For any central simple algebra B, let us define $\mathrm{K}(B)$ as the Grothendieck group of the category of finitely generated left B-modules. The group $\mathrm{K}(B)$ is infinite cyclic with a canonical generator given by the class of a (unique up to isomorphism) simple B-module.

Theorem 2.1 ([Qui73, §8, Theorem 4.1])**.** *Let X be the Severi-Brauer variety of a central simple algebra A. The group homomorphism*

$$\bigoplus_{i=0}^{\deg(A)-1} \mathrm{K}(A^{\otimes i}) \to \mathrm{K}(X),$$

[1]In its original formulation [Kar17b, Conjecture 1.1], Conjecture 1.2 only asserts there is an isomorphism in the case P is a Borel subgroup. However, to prove Conjecture 1.2 for all special parabolic subgroups of G it suffices to check the result holds for a particular choice of special parabolic subgroup P. These two forms of Conjecture 1.2 are then equivalent since a Borel subgroup is special.

mapping the class of a left $A^{\otimes i}$-module M to the class of $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$, is an isomorphism.

Note that if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects the decomposition of Theorem 2.1, is injective, and the image

$$\mathrm{K}(A^{\otimes i}) \subset \mathrm{K}(A_F^{\otimes i}) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A^{\otimes i})/\mathrm{ind}(A_F^{\otimes i})$. For $i \geq 0$, let us write $\zeta_X(i)$ for the tensor product (over $A^{\otimes i}$) of $\zeta_X^{\otimes i}$ by a simple $A^{\otimes i}$-module. This is a vector bundle of rank $\mathrm{ind}(A^{\otimes i})$ and $\zeta_X^{\otimes i}$ decomposes into a direct sum of $\deg(A^{\otimes i})/\mathrm{ind}(A^{\otimes i})$ copies of $\zeta_X(i)$.

A similar description is afforded to the rings $\mathrm{K}(X)$ for products $X = X_1 \times \cdots \times X_r$ of Severi-Brauer varieties:

Theorem 2.2 (cf. [Pey95, Corollary 3.2])**.** *Let $X = X_1 \times \cdots \times X_r$ be a product of Severi-Brauer varieties $X_1, ..., X_r$ corresponding to central simple algebras $A_1, ..., A_r$ respectively. Then the group homomorphism*

$$\bigoplus_{I < (\deg(A_1), ..., \deg(A_r))} \mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \to \mathrm{K}(X),$$

as $I = (i_1, ..., i_r)$ ranges over r-tuples of nonnegative integers, is an isomorphism. Here the class of a left $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$-module M is sent to the class $\zeta_{X_1}^{\otimes i_1} \boxtimes \cdots \boxtimes \zeta_{X_r}^{\otimes i_r} \otimes_{A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}} M$.

Similarly, if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects this decomposition, is injective, and the image

$$\mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \subset \mathrm{K}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})/\mathrm{ind}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F)$.

Given two products of Severi-Brauer varieties $X = X_1 \times \cdots \times X_r$ and $Y = Y_1 \times \cdots \times Y_r$, over possibly different fields F_1 and F_2 with $\dim(X_i) = \dim(Y_i)$ for every $1 \leq i \leq r$, let us identify $\mathrm{K}(X_{\overline{F_1}})$ with $\mathrm{K}(Y_{\overline{F_2}})$ via the isomorphism of Theorem 2.2. Let us also identify $\mathrm{K}(X)$ and $\mathrm{K}(Y)$ with their images in $\mathrm{K}(X_{\overline{F_1}}) = \mathrm{K}(Y_{\overline{F_2}})$. Note that we have $\mathrm{K}(X) = \mathrm{K}(Y)$ if and only if

$$\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \mathrm{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r})$$

for all integers $i_1, ..., i_r$, where $A_1, ..., A_r$ are the algebras associated to $X_1, ..., X_r$ and $B_1, ..., B_r$ are the algebras associated to $Y_1, ..., Y_r$.

The following statement shows that (unlike the coniveau filtration) the γ-filtration on $\mathrm{K}(X)$ is completely determined by $\mathrm{K}(X)$:

Theorem 2.3 ([IK99, Theorem 1.1 and Corollary 1.2])**.** *If $\mathrm{K}(X) = \mathrm{K}(Y)$, then $F_\gamma^i(X) = F_\gamma^i(Y)$ for all $i \geq 0$.*

3. Equivalence of the two conjectures

Let G be an affine algebraic group, let U be a non-empty open G-invariant subset of a G-representation V. If the fppf quotient U/G is representable by a scheme, and if U is a G-torsor over U/G, then U has the property that for any G-torsor H over an infinite field $F \supset k$, there is an F-point x of U/G so that H is isomorphic to the fiber of the morphism $U \to U/G$ over x, c.f. [Ser03, §5]. The generic fiber E of the quotient map $U \to U/G$ is called a *standard generic* G-torsor.

Example 3.1. If $G = \mathrm{SL}_n$, then G acts on $V = \mathrm{End}(k^n)$ with $\mathrm{GL}_n \subset V$ an open, G-invariant subset. The generic fiber $E = \mathrm{SL}_{n,k(\mathbb{G}_m)}$ of the quotient $\mathrm{GL}_n \to \mathrm{GL}_n/G = \mathbb{G}_m$ is a standard generic G-torsor.

A standard generic G-torsor E exists for any affine algebraic group G: one can take E to be the generic fiber of the quotient morphism $\mathrm{GL}_n \to \mathrm{GL}_n/G$ for any embedding $G \hookrightarrow \mathrm{GL}_n$.

Now assume G is a split semisimple algebraic group, with P a special parabolic subgroup of G, and E a standard generic G-torsor. Recall an algebraic group H over a field k is *special* if every H-torsor over any field extension of k is trivial. The quotient E/P is a *generic flag variety*, which is moreover *generically split*, meaning that E becomes trivial after scalar extension to the function field $k(E/P)$, c.f. [Kar18, Lemma 7.1].

Example 3.2. Let $G = \mathrm{SL}_n/\mu_m$, where m is a divisor of n. Then G acts on $X = \mathbb{P}^{n-1}$ and, if P is the stabilizer of a rational point in X, the quotient G/P is isomorphic to X. The parabolic P is special, it's conjugacy class is given by the subset of the Dynkin daigram of G corresponding to removing the first vertex, see [Kar18, §8].

If E is a standard generic G-torsor given as the generic fiber of a quotient map $U \to U/G$, then our identification of $G/P \cong X$ above shows that the generic flag variety E/P is a Severi-Brauer variety over the function field $k(U/G)$. The central simple $k(U/G)$-algebra associated to E/P is called a *generic central simple algebra of degree* n *and exponent* m. The index of such an algebra is equal to r where $n = rs$ is a factorization of n with r having the same prime factors as m and with s prime to m.

In [Kar17a], the first named author proves Conjecture 1.1 for the Severi-Brauer variety of a generic central simple algebra of degree n and exponent m and, as a Corollary obtained by analysis similar to Example 3.2 above, proves Conjecture 1.2 for split semisimple almost-simple algebraic groups of type A and C. In this section we prove an equivalence between Conjecture 1.1 and Conjecture 1.2 for algebraic groups of type AC similar to that obtained in [Kar17a] for a single Severi-Brauer variety and for a split semisimple almost-simple group of type A or of type C:

Theorem 3.3. *The following statements are equivalent:*

1. *Conjecture 1.1 holds for all* X*,*
2. *Conjecture 1.2 holds for all* G *of type* AC *and* P *given by removing the first vertex from each of the connected components of the Dynkin diagram of* G*,*
3. *Conjecture 1.2 holds for all* G *of type* AC *and arbitrary* P*,*
4. *Conjecture 1.2 holds for all* G *of type* AA *and arbitrary* P*.*

The proof is given below Lemma 3.6, after some preparation. It proceeds by showing (1) implies (2) implies (3) implies (4) implies (1). The most difficult part of the proof is in showing the last step, (4) implies (1). To do this, one realizes a product of Severi-Brauer varieties $X = X_1 \times \cdots \times X_r$ as a specialization of a generic flag variety E/P for a certain choice of split semisimple algebraic group G of type AA, standard generic G-torsor E, and special parabolic P. With mild hypotheses, one can show that this will prove the claim:

Lemma 3.4. *Let* G *be a split semisimple algebraic group of type* AA*,* E *a standard generic* G*-torsor, and* P *a special parabolic subgroup of* G*. Let* X *be a product of Severi-Brauer varieties such that* X *is a specialization of* E/P*. Assume the following conditions hold:*

1. $\mathrm{CH}(X)$ *is generated by Chern classes,*
2. *the canonical surjection* $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ *is an isomorphism,*
3. *the specialization* $\mathrm{K}(E/P) \to \mathrm{K}(X)$ *is an isomorphism.*

Then the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Proof. Since X is a specialization of E/P, there is a commutative diagram

(D)
$$\begin{array}{ccc} \mathrm{CH}(E/P) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(E/P) \\ \downarrow & & \downarrow \\ \mathrm{CH}(X) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(X) \end{array}$$

where the downward-pointing vertical arrows are specializations and the horizontal arrows are the canonical surjections.

In the diagram (D) above, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption and $\mathrm{CH}(X)$ is generated by Chern classes by assumption. Note that $\mathrm{CH}(E/P)$ is also generated by Chern classes, by [Kar18, Corollary 7.2 and Theorem 7.3]. Since the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism it follows the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is surjective.

The specialization $\mathrm{gr}_\tau \mathrm{G}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism: it fits into the commutative square below with the vertical arrows being specializations and the horizontal arrows being the canonical maps; the horizontal arrows are isomorphisms since the Chow rings $\mathrm{CH}(E/P)$ and $\mathrm{CH}(X)$ are generated by Chern classes, [Kar98, proof of Theorem 3.7]; the left-vertical arrow is an isomorphism since by Theorem 2.3 the isomorphism $\mathrm{K}(E/P) \to \mathrm{K}(X)$ induces a bijection $F^i_\gamma(E/P) \cong F^i_\gamma(X)$ for all i.

$$\begin{array}{ccc} \mathrm{gr}_\gamma \mathrm{K}(E/P) & \xrightarrow{\sim} & \mathrm{gr}_\tau \mathrm{G}(E/P) \\ \downarrow \wr & & \downarrow \\ \mathrm{gr}_\gamma \mathrm{K}(X) & \xrightarrow{\sim} & \mathrm{gr}_\tau \mathrm{G}(X) \end{array}$$

Hence the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is also an injection and therefore an isomorphism. It follows the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism as well, completing the proof. □

The problem is to find the correct G, P, and E that satisfy the conditions of Lemma 3.4. The naïve method, taking $E/P = E_1/P_1 \times \cdots \times E_r/P_r$ to be a product of generic flag varieties with each E_i/P_i having X_i as a specialization fails in at least one regard: the algebras associated to such an E/P are usually too unrelated. That is to say, the specialization in (3) of Lemma 3.4 will typically not be a surjection.

The following result of Nguyen, giving a description to the central simple algebras obtained from a G-torsor for split semisimple algebraic groups G of type AA, provides at least one resolution to this problem.

Theorem 3.5 ([CR15, Theorem A.1])**.** *Let $\Gamma = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a product of r general linear groups for some integers $n_1, ..., n_r$. Let C be a central subgroup of Γ, and write $G = \Gamma/C$. Let $\pi : G \to \Gamma/Z(\Gamma)$ be the natural projection. Then, for every field extension F of k, π_* identifies $H^1(F, G)$ with the set of isomorphism classes of r-tuples $(A_1, ..., A_r)$ of central simple F-algebras such that the degree of each A_i is $\deg(A_i) = n_i$, and $A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r}$ is split over F for every r-tuple of*

$$\mathscr{X}^*(Z(\Gamma)/C) = \{(m_1, ..., m_r) \in \mathbb{Z}^r \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \ \forall (\tau_1, ..., \tau_r) \in C\}.$$

To apply the theorem above to get the same description for the algebras associated to a G-torsor for a split semisimple algebraic group G of type AA, one notes that such a G is isomorphic to a quotient of a product $G_{sc} = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$ by a central subgroup C of G_{sc}. One can then use the quotient $G' = G^{red}/C$ of the reductive group $G^{red} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and the canonical inclusion $\iota : G \to G'$, taking into account that the induced map on cohomology $\iota_* : H^1(F, G) \to H^1(F, G')$ is a surjection (with trivial kernel).

It turns out, with the description given in Theorem 3.5, one has sufficient control to ensure the conditions of Lemma 3.4 hold (up to introducing some additional factors, which won't matter in the end).

Lemma 3.6. *Let $X_1, ..., X_r$ be a finite number of Severi-Brauer varieties corresponding to central simple k-algebras $A_1, ..., A_r$ and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ for all $1 \le i \le r$. For every r-tuple of nonnegative integers $I = (i_1, ..., i_r)$, write D_I for the underlying* *division algebra of the product $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ and write $Y_I = \mathrm{SB}(D_I)$ for the associated Severi-Brauer variety. Let $Z = X \times \prod_{I<(n_1,...,n_r)} Y_I$.*

In this setting, there exists a split semisimple algebraic group G of type AA *and a special parabolic P of G so that for any standard generic G-torsor E, the variety Z is a specialization of E/P and the specialization map $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ is an isomorphism.*

Proof. For every such r-tuple $I = (i_1, ..., i_r)$ we set $m_I := \mathrm{ind}(D_I)$ to be the index of D_I. The group

$$G_{sc} = \prod_{j=1}^{r} \mathrm{SL}_{n_j} \times \prod_{I<(n_1,...,n_r)} \mathrm{SL}_{m_I}$$

is split, semisimple, and simply connected of type AA. We consider the quotient $G := G_{sc}/S$, where S is the subgroup of the center of G_{sc} consisting of those elements

$$(x_1, ..., x_r, x_{(0,...,0)}, ..., x_{(n_1-1,...,n_r-1)})$$

satisfying the relation $x_{(i_1,...,i_r)} = x_1^{i_1} \cdots x_r^{i_r}$ (when identified with elements of $\mathbb{G}_m$). Let E be a standard generic G-torsor. We let

$$\sigma : G \to G_{ad}, \quad \pi_i : G_{ad} \to \mathrm{PGL}_{n_i}, \quad \pi_I : G_{ad} \to \mathrm{PGL}_{m_I}$$

be the canonical isogeny, projection to the ith factor for $i \leq r$, and projection to the factor corresponding to the r-tuple I respectively.

Let G^{red} be the reductive group

$$G^{red} = \prod_{j=1}^{r} \mathrm{GL}_{n_j} \times \prod_{I<(n_1,...,n_r)} \mathrm{GL}_{m_I}$$

and set $G' = G^{red}/S$. Let T be the kernel of the quotient $G^{red} \to G_{ad}$. We fix the isomorphism of the character group $\mathscr{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ that identifies the character with weights $(i_1, ..., i_n)$ with the element $(i_1, ..., i_n)$. The subgroup S above is defined so that the inclusion $\mathscr{X}^*(T/S) \to \mathscr{X}^*(T)$ identifies $\mathscr{X}^*(T/S)$ with the sublattice generated by those elements

$$(i_1, ..., i_r, -\delta_{I(0,...,0)}, ..., -\delta_{I(n_1-1,...,n_r-1)}),$$

where $I = (i_1, ..., i_r) < (n_1, ..., n_r)$ is an r-tuple. For any field extension F of k, the map $\sigma_* : H^1(F, G) \to H^1(F, G_{ad})$ factors through the map $H^1(F, G) \to H^1(F, G')$, induced by the inclusion of G into G'; this puts us in position to apply the description in Theorem 3.5 of the algebras $B_i := (\pi_i \circ \sigma)_*(E)$, $C_I := (\pi_I \circ \sigma)_*(E)$. In particular, our choice of S implies $B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r}$ is Brauer equivalent with $C_{(i_1,...,i_r)}$.

Again by Theorem 3.5, each of the algerbas A_i are specializations of the algebras B_i and, additionally, for every r-tuple $I = (i_1, ..., i_r)$ we have an equality

$$m_I = \mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \mathrm{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r})$$

since the underlying division algebra D_I of $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ is a specialization of C_I. The first claim then results from the fact the variety

$$\prod_{i=1}^{r} \mathrm{SB}(B_i) \times \prod_{I<(n_1,...,n_r)} \mathrm{SB}(C_I)$$

is isomorphic with E/P which has Z as a specialization. The second claim results from the description of the rings $\mathrm{K}(E/P)$ and $\mathrm{K}(Z)$ given in Theorem 2.2. □

And now for the proof:

Proof of Theorem 3.3. We show (1) implies (2). To start, let G be a group of type AC and E be a standard generic G-torsor over a field extension F of our base k. Let G_{ad} be the adjoint group of G; it is isomorphic to a product

$$G_{ad} = \prod_{i=1}^{n} G_i$$

with each G_i a simple adjoint group of type A or type C. We write $\sigma : G \to G_{ad}$ for the canonical isogeny from G to its adjoint and $\pi_i : G_{ad} \to G_i$ for the projection to the ith factor of G_{ad}.

From the n maps $\pi_i \circ \sigma$ with varying i, we obtain n central simple F-algebras given by the images of E under the pushforwards on Galois cohomology

$$(\pi_i \circ \sigma)_*(E) \in \mathrm{im}(H^1(F, G) \to H^1(F, G_i)).$$

Let X be the product of the Severi-Brauer varieties associated to the n algebras $(\pi_i \circ \sigma)_*(E)$. Then X is isomorphic to E/P, where P is a parabolic subgroup of G whose conjugacy class is given by the subset of the set of vertices of the Dynkin diagram of G obtained by excluding the first vertex of each of its connected components. That the parabolic P obtained in this way is special is a consequence of Lemma 3.8 below since, by [Kar18, §8], the group $\sigma(P)$ is special. The claim now follows from [Kar18, Corollary 7.2 and Theorem 7.3], which shows $\mathrm{CH}(X)$ is generated by Chern classes, allowing us to apply (1) to $X \cong E/P$.

(2) implies (3) is a consequence of [Kar17a, Lemma 4.2].

(3) implies (4) is obvious.

We finish by showing (4) implies (1). Let $X_1, ..., X_r$ be Severi-Brauer varieties over a field k, corresponding to central simple algebras $A_1, ..., A_r$ respectively, and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ be the degree of the algebra A_i. For every r-tuple of nonnegative integers $I = (i_1, ..., i_r)$ we write D_I for the underlying division algebra of the tensor product $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$. We write $Y_I := \mathrm{SB}(D_I)$ for the associated Severi-Brauer variety and $Z = X \times \prod_{I < (n_1, ..., n_r)} Y_I$ for the product of these varieties.

Let G and P be respectively an algebraic group of type AA and its special parabolic subgroup, obtained from Z as in Lemma 3.6. Let E be a standard generic G-torsor. By Lemma 3.7 below, to show the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism, it's sufficient to show $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism since the projection $Z \to X$ factors

$$Z \to X \times \prod_{I < (n_1, ..., n_{r-1}, n_r - 1)} Y_I \to \cdots \to X \times Y_{(0,...,0)} \to X$$

with each arrow a projective bundle. Finally, the arrow $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism by Lemma 3.4: $\mathrm{CH}(Z)$ is generated by Chern classes by repeated applications of the projective bundle formula and the assumption $\mathrm{CH}(X)$ is generated by Chern classes, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption, and the specialization $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ is an isomorphism. □

Lemma 3.7. *Assume Z is a projective bundle over a variety X. Then the canonical epimorphism $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism if, and only if, the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism.*

Proof. The pullback along the projection $Z \to X$ gives a commuting diagram

$$\begin{array}{ccc} \mathrm{CH}(Z) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(Z) \\ \uparrow & & \uparrow \\ \mathrm{CH}(X) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(X) \end{array}$$

with both vertical arrows injections. It follows if the top-horizontal arrow is an isomorphism, then the bottom-horizontal arrow is an isomorphism.

The converse follows from the projective bundle formula: the groups $\mathrm{CH}(Z)$ and $\mathrm{gr}_\tau \mathrm{G}(Z)$ are direct sums of several copies of of the groups $\mathrm{CH}(X)$ and $\mathrm{gr}_\tau \mathrm{G}(X)$ respectively, and the coniveau epimorphism respects this direct sum decomposition. □

Lemma 3.8. *Let G be a split semisimple algebraic group over a field F, and $\sigma : G \to G_{ad}$ the canonical isogeny with kernel C, the center of G. If P is a parabolic subgroup of G such that the image $\sigma(P)$ is special, then P is special.*

Proof. Let L be a Levi subgroup of P. By [Kar18, §3], P is special if and only if L is special. Since G is a split reductive group, P is also a split reductive group so that, by [Kar18, Theorem 2.1], L is special if and only if the semisimple commutator $L' \subset L$ is special. Similarly, $\sigma(P)$ is special if and only if $\sigma(L)'$ is special. Thus the proof of the lemma can be reduced to the following statement: if L' is a split semisimple algebraic group and $L' \to \sigma(L)'$ is an isogeny with $\sigma(L)'$ split, semisimple, and special, then L' is special. The result then follows from the fact a split semisimple algebraic group is special if and only if it is a product of special linear or symplectic groups and all such groups are simply connected. □

We conclude this section with some remarks on, and special cases of, Conjectures 1.1 and 1.2.

Remark 3.9. One can construct a large class of products X of Severi-Brauer varieties which satisfy the condition $\mathrm{CH}(X)$ is generated by Chern classes. To do so, let $G = \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r}$ for some $n_1, .., n_r \geq 2$; let $A_1, ..., A_r$ be the central simple algebras associated to a standard generic G-torsor; let X be the product of the associated Severi-Brauer varieties. By [Kar18, Theorem 7.3], $\mathrm{CH}(X)$ has the desired property.

One can extend this class by base change: it's possible to lower the index of any tensor product $A = A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ by extending the base to the function field of any generalized Severi-Brauer variety of A. The new variety X obtained from these algebras also has the property $\mathrm{CH}(X)$ is generated by Chern classes, [Kar98, Theorem 3.7]. This procedure can be repeated indefinitely.

In fact, to prove Conjecture 1.1 for all products of Severi-Brauer varieties, it suffices to prove Conjecture 1.1 for the varieties obtained by the above procedure (one can even restrict to the class whose construction involves the function field of *usual* Severi-Brauer varieties only); to go from the above case to the general case, one can use the specialization argument as in Theorem 3.3.

Example 3.10 ($\mathrm{A}_1 \times \mathrm{A}_1$ and $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$)**.** In small rank cases, one can check Conjecture 1.2 for G of type AA by hand.

For G as in Conjecture 1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1$ one can observe: for any projective homogeneous variety X of dimension less or equal 2, the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism, cf. [CM06, Proposition 4.4].

For G as in Conjecture 1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, one can proceed by cases. If G is a product of groups of smaller rank, then [Kar17b, Proposition 4.1] proves the claim. Otherwise, G is a quotient of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ by the diagonal of the center $\mu_2 \times \mu_2 \times \mu_2$ or by the subgroup generated by the partial 2-diagonals. In the first case, the corresponding generic flag variety is a product $C \times C \times C$ of a fixed conic C and the claim follows. In the second case, the corresponding generic flag variety is a product $X = C_1 \times C_2 \times C_3$ where each C_i is the conic of a quaternion algebra Q_i; here the sum of the classes $[Q_1] + [Q_2] + [Q_3]$ is trivial in the Brauer group. Since X is a projective bundle over any two of the factors this proves the result by Lemma 3.7.

Example 3.11. Conjecture 1.2 holds for $G = \mathrm{SL}_n/\mu_m$ by [Kar17a, Theorem 1.1] and for products of such groups by [Kar17b, Proposition 4.1]. From this, one can show Conjecture 1.1 holds for products $X = X_1 \times \cdots \times X_r$ satisfying the following conditions:

(1) for each $1 \leq i \leq r$ there is a prime p_i so that the algebra A_i associated to the variety X_i has index $p_i^{n_i}$ and exponent $p_i^{m_i}$ for some integers $n_i \geq m_i \geq 1$,
(2) the algebras A_i satisfy $\mathrm{ind}(A_i^{\otimes p_i^{m_i-1}}) = \mathrm{ind}(A_i)/p_i^{m_i-1}$,
(3) the algebras A_i are disjoint in the sense there are equalities

$$\mathrm{ind}(A_1^{\otimes i_r} \otimes \cdots \otimes A_r^{i_r}) = \mathrm{ind}(A_1^{\otimes i_1}) \cdots \mathrm{ind}(A_r^{\otimes i_r})$$

for all integers $i_1, ..., i_r$.

To see this, one may assume that all A_i are division algebras and use Lemma 3.4. Property (2) allows one to realize such an X as a specialization of E/P where E is a standard generic $G = \prod_{1 \leq i \leq r} \mathrm{SL}_{p_i^{n_i}}/\mu_{p_i^{m_i}}$-torsor and $P \subset G$ is a special parabolic subgroup whose conjugacy class can be obtained by removing the first vertex from each of the connected components of the Dynkin diagram of G. The canonical map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ for this E/P is an isomorphism, as explained above. Now property (3), [Kar17b, Lemma 4.3], and Theorem 2.3 show the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism.

Appendix A. Algebras with level 1

In this appendix we introduce the level of a central simple k-algebra. The level is a nonnegative integer that measures, roughly speaking, how far away the algebra is from having its index equal to its exponent. It's related to, and depends on, the reduced behavior of the primary components of the algebra as defined in [Kar98]. The same concept was considered in [Bae15], there as the length of a reduced sequence obtained from the reduced behavior of a p-primary algebra for a prime p; the length of this reduced sequence as defined by Baek is equal to the level of the p-primary algebra as defined here.

It turns out the level of a central simple algebra A can be used to obtain detailed information on λ-ring generators for the Grothendieck ring of the Severi-Brauer variety X of A, see Lemma A.6. A particular consequence of this is that the subring of $\mathrm{CH}(X)$ which is generated by Chern classes has an explicit and small set of generators that can be helpful for computational purposes. Using this more refined information based on the level, we're able to generalize the results of [Kar17a] to prove the main result, Theorem A.15, that the K-theory coniveau epimorphism is an isomorphism for Severi-Brauer varieties whose Chow ring is generated by Chern classes and whose associated central simple algebra has level 1.

Throughout this appendix we work with a fixed prime p and we continue to work over the fixed but arbitrary field k. We write $v_p(-)$ for the p-adic valuation. We've relegated some computations needed in this section to Appendix B.

Recall, the *reduced behavior* of an algebra A with index $\mathrm{ind}(A) = p^n$ and exponent $\exp(A) = p^m$, $0 < m \leq n$, is defined to be the following sequence of p-adic orders of increasing p-primary tensor powers of A:

$$r\mathcal{B}eh(A) = \left(v_p(\mathrm{ind}(A^{\otimes p^i}))\right)_{i=0}^{m} = \left(v_p(\mathrm{ind}(A)), v_p(\mathrm{ind}(A^{\otimes p})), ..., v_p(\mathrm{ind}(A^{\otimes p^m}))\right).$$

The reduced behavior of A is strictly decreasing; it starts with $v_p(\mathrm{ind}(A)) = n$ and ends with $v_p(\mathrm{ind}(A^{\otimes p^m})) = 0$.

Definition A.1. A is said to have *level* l, abbreviated $\mathrm{lev}(A) = l$, if there exist exactly l distinct integers $i_1, ..., i_l \geq 1$ with $v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1$ for every $1 \leq k \leq l$. If no such integers exist, A is said to have level 0. An arbitrary central simple algebra B, not necessarily p-primary, is said to have *level* l if l is the maximum

$$l = \max_{q \text{ prime}} \{\mathrm{lev}(B_q)\}$$

of the levels of the q-primary components B_q of B.

Example A.2. A central simple algebra A has level 0, i.e. $\operatorname{lev}(A) = 0$, if and only if the index and exponent of A coincide, $\operatorname{ind}(A) = \exp(A)$.

Example A.3. If A is a generic algebra of degree p^n and exponent p^m with $m < n$, in the sense of Example 3.2, then the level of A is 1, i.e. $\operatorname{lev}(A) = 1$. The reduced behavior for this algebra is

$$r\mathcal{B}eh(A) = \big(v_p(\operatorname{ind}(A)), v_p(\operatorname{ind}(A^{\otimes p})), ..., v_p(\operatorname{ind}(A^{\otimes p^m}))\big) = (n, n-1, ..., n-m+1, 0).$$

To see this, note that with a large enough field extension F of k one may find a central division F-algebra B with index p^n, exponent p^m, and reduced behavior $r\mathcal{B}eh(B) = (n, n-1, ..., n-m+1, 0)$, [Kar98, Lemma 3.10]. Since B is a specialization of A it follows

$$p^{n-i} \geq \operatorname{ind}(A^{\otimes p^i}) \geq \operatorname{ind}(B^{\otimes p^i}) = p^{n-i}$$

for $i = 0, \dots, m-1$, so that equalities hold throughout.

We make the following definition for notational convenience.

Definition A.4. The *Chern subring* of a smooth variety X, denoted $\operatorname{CS}(X)$, is the subring of $\operatorname{CH}(X)$ which is generated by all Chern classes of elements of $\operatorname{K}(X)$.

Proposition A.5. *Let X be the Severi-Brauer variety of a central simple algebra A with $\operatorname{ind}(A) = p^n$ and $\operatorname{lev}(A) = r$. Then $\operatorname{CS}(X)$ is generated, as a ring, by the Chern classes of $r+1$ sheaves on X. Namely, the sheaves whose Chern classes generate $\operatorname{CS}(X)$ are:*

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \dots,\ \zeta_X(p^{i_r}),$$

where $1 \leq i_1 < \dots < i_r$ are the r distinct integers with $v_p(\operatorname{ind}(A^{\otimes p^{i_k}})) < v_p(\operatorname{ind}(A^{\otimes p^{i_k-1}})) - 1$.

Proof. It suffices to show that $\operatorname{K}(X)$ is generated by the classes of

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \dots,\ \zeta_X(p^{i_r})$$

as a λ-ring; this is because Chern classes of λ-operations of an element of $\operatorname{K}(X)$ are certain universal polynomials in the Chern classes of this element. This is done in the next lemma. □

Lemma A.6. *Let X be the Severi-Brauer variety of a central simple algebra A with $\operatorname{ind}(A) = p^n$ and $\operatorname{lev}(A) = r$. Then $\operatorname{K}(X)$ is generated, as a λ-ring, by $r+1$ elements. Namely, the sheaves whose classes generate $\operatorname{K}(X)$ are:*

$$\zeta_X(1),\ \zeta_X(p^{i_1}),\ \dots,\ \zeta_X(p^{i_r}),$$

where $1 \leq i_1 < \dots < i_r$ are the r distinct integers with $v_p(\operatorname{ind}(A^{\otimes p^{i_k}})) < v_p(\operatorname{ind}(A^{\otimes p^{i_k-1}})) - 1$.

Proof. Since the pullback $\pi^* : \operatorname{K}(X) \to \operatorname{K}(X_L)$ to a splitting field L of A is injective, we can work, instead of $\operatorname{K}(X)$ itself, with its image in $\operatorname{K}(X_L)$. We'll write ξ to denote the class of $\mathcal{O}(-1)$ in $\operatorname{K}(X_L)$. By the comments under Theorem 2.1 we have $\pi^*(\zeta_X(i)) = \operatorname{ind}(A^{\otimes i})\xi^i$. It follows that the elements $\operatorname{ind}(A^{\otimes i})\xi^i$ with $i \geq 0$ generate $\operatorname{K}(X)$ as an abelian group.

The λ-operations of any multiple of ξ^i are easy to compute:

$$\lambda^j(d\xi^i) = \binom{d}{j}\xi^{ij} \quad \text{for any } i, j, d \geq 0.$$

Let us first show that the elements $\operatorname{ind}(A^{\otimes p^j})\xi^{p^j}$ $(j \geq 0)$ generate $\operatorname{K}(X)$ as a λ-ring. Since the λ-subring generated by these elements contains powers of $\operatorname{ind}(A)\xi = p^n\xi$, we only need to check that, for every $i \geq 1$, this subring contains an integer multiple of ξ^i whose coefficient has p-adic valuation equal $v_p(\operatorname{ind}(A^{\otimes i}))$. For this, given any $i \geq 1$, we write $i = p^j s$ with $j \geq 0$ and s prime-to-p. We set $p^v := \operatorname{ind}(A^{\otimes i}) = \operatorname{ind}(A^{\otimes p^j})$. Write further $s = s_0p^v + s_1$ with $0 \leq s_1 < p^v$ and $s_0 \geq 0$.

Then we have $\lambda^{p^v}(p^v\xi^{p^j}) = \xi^{p^j p^v}$ and $\lambda^{s_1}(p^v\xi^{p^j})$ is a multiple of $\xi^{p^j s_1}$ with p-adic valuation of the (binomial) coefficient of this multiple equal p^v, see [Kar98, Lemma 3.5]. The claim we are checking follows.

It remains to show if $v_p(\mathrm{ind}(A^{\otimes p^j})) \geq v_p(\mathrm{ind}(A^{\otimes p^{j-1}})) - 1$ for some $j \geq 1$, then the generator $\mathrm{ind}(A^{\otimes p^j})\xi^{p^j}$ can be omitted. Let us set $p^v := \mathrm{ind}(A^{\otimes p^{j-1}})$. If $v = 0$, then we get ξ^{p^j} as a pth power of $\xi^{p^{j-1}} = \mathrm{ind}(A^{\otimes p^{j-1}})\xi^{p^{j-1}}$. For $v > 0$, we consider the λ-operation $\lambda^p(p^v\xi^{p^{j-1}})$ which is a multiple of ξ^{p^j} with p-adic valuation of its coefficient equal $v - 1 \leq v_p(\mathrm{ind}(A^{\otimes p^j}))$. □

To systematically study the relations between the Chern classes of the sheaves appearing in Proposition A.5, we introduce:

Definition A.7. Let A be a central simple algebra and X the Severi-Brauer variety of A. We write $\mathrm{CT}(i_1, ..., i_r; X)$ for the graded subring of $\mathrm{CS}(X) \subset \mathrm{CH}(X)$ generated by the Chern classes of the sheaves $\zeta_X(i_1), ..., \zeta_X(i_r)$.

Proposition A.8. *Let X be the Severi-Brauer variety of a central simple algebra A. Then, for any $i > 0$, $\mathrm{CT}(i; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Moreover, for $0 \leq j < \deg(A)$ the group $\mathrm{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ is additively generated by*

$$\tau_i(j) := c_{p^v}(\zeta_X(i))^{s_0} c_{s_1}(\zeta_X(i))$$

where p^v is the largest power of p dividing $\mathrm{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \leq s_1 < p^v$.

Proof. By first extending to a prime-to-p extension (which is an injection when $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ has $\mathbb{Z}_{(p)}$-coefficients) that splits the prime-to-p components of A, we can assume A is p-primary. We continue by reducing to the case $i = 1$.

Lemma A.9. *Let X be the Severi-Brauer variety of a central simple algebra A, and let Y be the Severi-Brauer variety of $A^{\otimes i}$. Then there is a functorial surjection*

$$\mathrm{CT}(1; Y) \twoheadrightarrow \mathrm{CT}(i; X).$$

Proof. Let

$$X \to X^{\times i} \to Y$$

be the composition of the diagonal embedding and the twisted Segre embedding. The corresponding maps on Grothendieck groups can be determined by moving to a splitting field L of X. There is a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}(Y_L) & \longrightarrow & \mathrm{K}(X_L^{\times i}) & \longrightarrow & \mathrm{K}(X_L) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{K}(Y) & \longrightarrow & \mathrm{K}(X^{\times i}) & \longrightarrow & \mathrm{K}(X) \end{array}$$

defined so that under the top-horizontal maps we have

$$\mathcal{O}_{Y_L}(-1) \mapsto \mathcal{O}_{X_L}(-1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_L}(-1) \mapsto \mathcal{O}_{X_L}(-i).$$

Thus, the class of $\zeta_Y(1)$ on Y is mapped to the class of $\zeta_X(i)$ on X.

So under the composition of the diagonal $X \to X^{\times i}$ and the twisted Segre embedding $X^{\times i} \to Y$, there is a surjection $\mathrm{CT}(1; Y) \twoheadrightarrow \mathrm{CT}(i; X)$ induced by the pullback $\mathrm{CH}(Y) \to \mathrm{CH}(X)$. □

Next we reduce to the case our algebra is division. Let D be the underlying division algebra of A, and Y the Severi-Brauer variety of D. Fix an embedding $Y \to X$ so that, over a splitting field of both, the inclusion is as a linear subvariety. The pullback

$$\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(Y) \otimes \mathbb{Z}_{(p)}$$

is an isomorphism in degrees where both groups are nonzero. If the claim is true for $\mathrm{CH}(Y)\otimes\mathbb{Z}_{(p)}$ then, since the pullback is functorial for Chern classes, we find $\mathrm{CT}^j(1;X)\otimes\mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module of rank 1 in degrees $0\leq j<\deg(D)$. That this holds is due to [Kar17a, Proposition 3.3], where it's shown $\mathrm{CT}(1;X)$ is free if A is division. This will serve as the base case for an induction proof.

In an arbitrary degree j between $\deg(D)\leq j<\deg(A)$, we assume the claim is true for all degrees $0\leq k<j$. It suffices to show the multiplication by $\tau_1(p^v)=c_{p^v}(\zeta_X(1))$ map

$$\mathrm{CT}^{j-p^v}(1;X)\otimes\mathbb{Z}_{(p)}\to\mathrm{CT}^j(1;X)\otimes\mathbb{Z}_{(p)}$$

is surjective and, by Nakayama's Lemma, we can do this modulo p. Any element of $\mathrm{CT}^j(1;X)$ is a sum of monomials of the form $\tau_1(j-p^v)c_{i_1}^{n_1}\cdots c_{i_r}^{n_r}$ with $c_i=c_i(\zeta_X(1))$. We claim any such monomial which is not $\tau_1(j)=\tau_1(j-p^v)\tau_1(p^v)$ is congruent to 0 modulo p.

Indeed, if such a monomial was divisible by c_{i_1},c_{i_2} then without loss of generality we can assume $v_p(i_2)\leq v_p(i_1)<v$. By [Kar17a, Proposition 3.5] there is a field F finite over the base so that $v_p\mathrm{ind}(A_F)=v_p(i_1)$, and $c_{i_1}=\pi_*(x)$ for an element x of $\mathrm{CH}(X_F)\otimes\mathbb{Z}_{(p)}$ and where $\pi:X_F\to X$ is the projection. Using the projection formula we find

$$c_{i_1}c_{i_2}=\pi_*(x)c_{i_2}=\pi_*(x\pi^*(c_{i_2})).$$

By Lemma A.10 below, it follows $\pi^*(c_{i_2})$ is divisible by p which proves the claim.

To see the generators are as claimed for $i=1$, one can compute the degrees of the images of the Chern classes of $\zeta_X(1)$ over an algebraic closure; for the other i, one can use Lemma A.9. □

Lemma A.10. *Let X be the Severi-Brauer variety of a central simple algebra A with $\mathrm{ind}(A)=p^v$. Let F be a field with $p^{v-s}=\mathrm{ind}(A_F)<\mathrm{ind}(A)=p^v$ and let $\pi:X_F\to X$ be the projection. Then*

$$\pi^*(c_j(\zeta_X(1))=0\pmod p$$

for all j not divisible by p^v.

Proof. We have $\pi^*(\zeta_X(1))=\zeta_{X_F}(1)^{\oplus p^s}$ with $p^s=\mathrm{ind}(A)/\mathrm{ind}(A_F)$. By functorality we have

$$\pi^*(c_j(\zeta_X(1)))=c_j(\zeta_{X_F}(1)^{\oplus p^s}).$$

We're going to compute the total Chern polynomial of $\zeta_{X_F}(1)^{\oplus p^s}$ modulo p. If F splits A then $c_t(\zeta_{X_F}(1)^{\oplus p^s})=(1-h)^{p^s}=1\pm h^{p^s}\pmod p$ where h is the class of a hyperplane in $\mathrm{CH}(X_F)$. Otherwise $v\neq s$ and we have

$$c_t(\zeta_{X_F}(1)^{\oplus p^s})=c_t(\zeta_{X_F}(1))^{p^s}=(1+c_1t+\cdots+c_{p^{v-s}}t^{p^{v-s}})^{p^s}$$

with $c_i=c_i(\zeta_{X_F}(1))$. Using the multinomial formula, the latter expression can be rewritten

$$(1+c_1t+\cdots+c_{p^{v-s}}t^{p^{v-s}})^{p^s}=1+\sum_{j=1}^{p^v}\left(\sum_{\substack{|I|=p^s\\ i_1+2i_2+\cdots+p^{v-s}i_{p^{v-s}}=j}}\binom{p^s}{i_0,i_1,...,i_{p^{v-s}}}c_1^{i_1}\cdots c_{p^{v-s}}^{i_{p^{v-s}}}\right)t^j.$$

Here the notation means $\binom{n}{a_0,\ldots,a_i}=\frac{n!}{a_0!\cdots a_i!}$ and $I=(i_0,...,i_{p^{v-s}})$ is a tuple of nonnegative integers with $|I|=i_0+\cdots+i_{p^{v-s}}$.

By Lemma B.3, p divides all of the coefficients $\binom{p^s}{i_0,\ldots,i_{p^{v-s}}}$ except when p^s divides one of $i_0,...,i_{p^{v-s}}$. We are left to show $c_{i_k}^{p^s}=0$ modulo p for any $k=0,...,p^{v-s}-1$. Using [Kar17a, Proposition 3.5], we can find a finite field extension E/F lowering the index of A_F and such that $c_{i_k}=\rho_*(x)$ for some x in $\mathrm{CH}(X_E)\otimes\mathbb{Z}_{(p)}$ and for $\rho:X_E\to X_F$ the projection. The projection formula then gives

$$c_{i_k}^{p^s}=\rho_*(x(\rho^*\rho_*(x))^{p^s-1})=0\pmod p$$

since $\rho^*\rho_* = [E : F]$. □

Corollary A.11. *Let A be a central simple algebra and X its associated Severi-Brauer variety. The classes $\tau_i(j)$ of $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ satisfy the relations:*

(1) *for all $i \geq 1$, we have $\tau_i(0) = 1$,*
(2) *for any $j \geq 0$, we have $\tau_i(p^v)\tau_i(j) = \tau_i(p^v j)$, where $v = v_p(\mathrm{ind}(A^{\otimes i}))$,*
(3) *for any integers $a_1, ..., a_{p^v} \geq 0$, there is a relation*

$$\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}} = \alpha\tau_i(a_1 + 2a_2 + \cdots + p^v a_{p^v})$$

for some α in $\mathbb{Z}_{(p)}$ with

$$v_p(\alpha) = \begin{cases} 0 & \text{if} \quad v = 0 \\ \sum_{k=1}^{p^v}(v - v_p(k))a_k & \text{if} \quad v > 0 \quad \text{and} \quad j = 0 \pmod{p^v} \\ v_p(r) - v + \sum_{k=1}^{p^v}(v - v_p(k))a_k & \text{if} \quad v > 0 \quad \text{and} \quad j \neq 0 \pmod{p^v} \end{cases}$$

where we write $j = a_1 + 2a_2 + \cdots + p^v a_{p^v}$ and $0 \leq r < p^v$ is the remainder in the division of j by p^v.

Proof. We remark that the definition of the classes $\tau_i(j)$ makes sense for any integer $j \geq 0$ but when $j > \deg(A)$ these classes are 0. For simplifications below, we don't put any upper bound on the value j may have.

The relation (1) is obvious from the definition. The relation (2) is also clear from the definition. So we're left proving the complicated relation (3). To do this, we pullback, to a splitting field L, the left and right side of the equation in (3) and compare p-adic valuations of their coefficients on the element h^j where h is the class of a hyperplane over L. Some immediate observations for the following: we can assume j isn't larger than the dimension of X and we can assume $v > 0$; otherwise the claim is trivial.

The pullback of $\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}}$ can be written βh^j where

$$v_p(\beta) = \sum_{k=1}^{p^v}(v - v_p(k) + v_p(i)k)a_k.$$

Similarly, the pullback of $\tau_i(a_1 + \cdots + p^v a_{p^v})$ can be written γh^j with

$$v_p(\gamma) = \begin{cases} v_p(i)p^v s_0 & \text{if} \quad j = 0 \pmod{p^v} \\ v_p(i)p^v s_0 + v - v_p(s_1) + v_p(i)s_1 & \text{if} \quad j \neq 0 \pmod{p^v} \end{cases}$$

where $j = s_0 p^v + s_1$ and $0 \leq s_1 < p^v$. Since $v_p(\gamma) \geq v_p(\beta)$ by Proposition A.8, the result follows by subtracting. □

Lemma A.12. *Let A be a central simple algebra with $\mathrm{ind}(A) = p^n$ and $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Let X be the Severi-Brauer variety of A. Then, for any pair of integers i, j with $0 \leq i \leq j \leq m$, the total Chern polynomial*

$$c_t(\zeta_X(p^j))^{p^{n_i - n_j - (j-i)}} = 1 + \sum_{k=1}^{p^{n_i - (j-i)}} \beta_k \tau_{p^j}(k)t^k$$

is a polynomial with coefficients in $\mathrm{CT}(p^i; X) \otimes \mathbb{Z}_{(p)}$.

Moreover, the p-adic valuation of the coefficient β_k equals

$$v_p(\beta_k) = \begin{cases} n_i - n_j - (j-i) - v_p(k/p^{n_j}) & \text{if} \quad k = 0 \pmod{p^{n_j}} \\ n_i - n_j - (j-i) & \text{if} \quad k \neq 0 \pmod{p^{n_j}}. \end{cases}$$

Proof. We identify $\mathrm{K}(X)$ with its image in $\mathrm{K}(X_L)$ for a splitting field L of X. We write ξ for the class of $\mathcal{O}(-1)$ in $\mathrm{K}(X_L)$. Then the class of $\zeta_X(p^i)$ is identified with $p^{n_i}\xi^{p^i}$ and the class of $\zeta_X(p^j)$ is identified with $p^{n_j}\xi^{p^j}$. We have

$$\lambda^{p^{j-i}}(p^{n_i}\xi^{p^i}) = \binom{p^{n_i}}{p^{j-i}}\xi^{p^j}.$$

It follows that

$$\begin{aligned} c_t(p^{n_i-(j-i)}\xi^{p^j}) &= c_t(p^{n_i-(j-i)-n_j}(p^{n_j}\xi^{p^j})) \\ &= c_t(\zeta_X(p^j))^{p^{n_i-n_j-(j-i)}} \\ &= (1+\tau_{p^j}(1)t+\cdots+\tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} \end{aligned}$$

is a polynomial with coefficients contained in $\mathrm{CT}(p^i;X)\otimes\mathbb{Z}_{(p)}$. This proves the first claim.

To prove the second claim, we write

$$= (1+\tau_{p^j}(1)t+\cdots+\tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} = 1+\sum_{k=1}^{p^{n_i-(j-i)}} \beta_k\tau_{p^j}(k)t^k$$

using Proposition A.8. Explicitly there are equalities

$$\beta_k\tau_{p^j}(k) = \sum_I \binom{p^{n_i-(j-i)-n_j}}{I}\tau_{p^j}^I$$

where the sum runs over tuples $I=(a_0,...,a_{p^{n_j}})$ such that $a_0+\cdots+a_{p^{n_j}} = p^{n_i-(j-i)-n_j}$ and $a_1+2a_2+\cdots+p^{n_j}a_{p^{n_j}}=k$; here we're using the notation

$$\binom{p^{n_i-(j-i)-n_j}}{I} = \binom{p^{n_i-(j-i)-n_j}}{a_0,...,a_{p^{n_j}}} = \frac{p^{n_i-(j-i)-n_j}!}{a_0!\cdots a_{p^{n_j}}!} \quad\text{and}\quad \tau_{p^j}^I = \tau_{p^j}(0)^{a_0}\tau_{p^j}(1)^{a_1}\cdots\tau_{p^j}(p^{n_j})^{a_{p^{n_j}}}$$

for a tuple $I=(a_0,...,a_{p^{n_j}})$. Thus

$$v_p(\beta_k) = v_p\left(\sum_I\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I\right) \geq \min\left\{v_p\left(\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I\right)\right\}$$

where α_I is the coefficient in $\tau_{p^j}^I = \alpha_I\tau_{p^j}(k)$ from Corollary A.11. In fact, the above inequality is an equality if there is a unique minimum over the given tuples I. The p-adic valuation of any coefficient $\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I$ can be found using Corollary A.11 and Lemma B.2; the p-adic valuation of any coefficient $\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I$ can also be bounded below using Corollary A.11 and Lemma B.3. With this bound, one can show there is a unique minimum among the $v_p(\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I)$: set $s=n_i-(j-i)$ and $r=n_j$ in Lemma B.4. Finally, using Lemma B.2 to compute the valuation explicitly and using Lemma B.5, setting $s=n_i-(j-i)$ and $r=n_j$, shows the p-adic valuation of β_k is as claimed. ☐

The lemma above provides a collection of numbers β_k with $\beta_k\mathrm{CT}^k(p^j;X)\subset\mathrm{CT}^k(1;X)$. Using a technique developed in [Kar17a], we can reduce the size of the β_j further. We assume A is a division algebra in the following as this is the only case we will need.

Corollary A.13. *Let A be a division algebra with $\mathrm{ind}(A)=p^n$ and $r\mathcal{B}eh(A)=(n_0,...,n_m)$. Let X be the Severi-Brauer variety of A. Pick an integer $0\leq j\leq m$, and let $0\leq i\leq p^n-1$ be a second integer.*

There exists a number α_i in $\mathbb{Z}_{(p)}$ so that $\alpha_i \tau_{p^j}(i)$ is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. *Moreover, the p-adic valuation of the α_i we find equals*

$$v_p(\alpha_i) = \begin{cases} n - j - n_j & \text{if} \quad 1 \leq i \leq p^{n_j} \\ n - j - n_j - \lfloor \log_p(i/p^{n_j}) \rfloor & \text{if} \quad p^{n_j} < i \leq p^{n-j} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let L be a maximal subfield of A, of degree p^n over the base, and let N be the image of the pushforward $\pi_* : \mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ along the projection $\pi : X_L \to X$. By [Kar17a, Proposition 3.5], the image N is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. Recall also the pullback π^* followed by the pushforward π^* is multiplication by p^n, the degree of L over the base. The proof of the corollary mimics that of [Kar17a, Proposition 3.12]; the idea of the proof is to use the explicit bounds of Lemma A.12 and the projection formula to get the result for any i. Note that the claim is trivial for $j = 0$ (or we can just set $\alpha_i = 1$ in this case) so, throughout the proof, it's safe to assume $j > 0$.

We first show, for $i \leq p^{n-j}$ and using β_i for the coefficient such that $\beta_i \mathrm{CT}^i(p^j; X) \subset \mathrm{CT}^i(1; X)$ found in Lemma A.12, that $p^{v_p(\beta_i)} \tau_{p^j}(i)$ is in the image of the map π_*. Write $i = s_0 p^{n_j} + s_1$ with $0 \leq s_1 < p^{n_j}$. The image of $\tau_{p^j}(i)$ in $\mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)}$ is equal, up to prime-to-p parts, to

$$\pi^*(\tau_{p^j}(i)) = \begin{cases} p^{ij} h^i & \text{if} \quad s_1 = 0 \\ p^{ij + n_j - v_p(s_1)} h^i & \text{if} \quad s_1 > 0. \end{cases}$$

By Lemma A.12, the multiple $\beta_i \tau_{p^j}(i)$ has image, up to prime-to-p parts,

$$\pi^*(\beta_i \tau_{p^j}(i)) = p^{n + (i-1)j - v_p(i)} h^i$$

regardless of s_1. Thus,

$$p^{v_p(\beta_i)} \tau_{p^j}(i) = \frac{1}{p^n} \pi_* \pi^*(p^{v_p(\beta_i)} \tau_{p^j}(i)) = \pi_*(\frac{1}{p^n}(\pi^*(p^{v_p(\beta_i)} \tau_{p^j}(i)))) = \pi_*(p^{(i-1)j - v_p(i)} h^i).$$

Since $(i-1)j - v_p(i) \geq 0$, we find $p^{v_p(\beta_i)} \tau_{p^j}(i)$ is in N as claimed.

Now let i be an integer with $1 \leq i \leq p^n - 1$ and set $\ell = \lfloor \log_p(i/p^{n_j}) \rfloor$. To get the bounds on the p-adic valuation in the corollary statement, we work in cases. We first assume $\ell \geq n - j - n_j$ or equivalently $i \geq p^{n-j}$. By the above and Lemma A.12, we can find an element x of $\mathrm{CH}(X_L)$ with

$$\pi_*(x) = \tau_{p^j}(p^{n-j}).$$

Set $k = i - p^{n-j}$. Then, using (2) and (3) of Corollary A.11,

$$\tau_{p^j}(i) = \tau_{p^j}(p^{n_j})^{n-j-n_j} \tau_{p^j}(k) = \tau_{p^j}(p^{n-j}) \tau_{p^j}(k) = \pi_*(x) \tau_{p^j}(k) = \pi_*(x \pi^*(\tau_{p^j}(k))).$$

By [Kar17a, Proposition 3.5], it follows $\tau_{p^j}(i)$ is contained in $N \subset \mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$ for all $i \geq p^{n-j}$.

For the other i, we act similarly. If $p^{n_j} < i \leq p^{n-j}$ then set $k = i - p^{n_j + \ell}$. Then there is a (different) element x with $\pi_*(x) = p^r \tau_{p^j}(p^{\ell + n_j})$ where $r = v_p(\beta_{p^{\ell + n_j}})$. Then

$$p^r \tau_{p^j}(i) = p^r \tau_{p^j}(p^{n_j})^{\ell} \tau_{p^j}(k) = p^r \tau_{p^j}(p^{\ell + n_j}) \tau_{p^j}(k) = \pi_*(x) \tau_{p^j}(k) = \pi_*(x \pi^*(\tau_{p^j}(k)))$$

and the claim follows as before.

For the remaining i, when $i \leq p^{n_j}$, the claim is actually immediate from Lemma A.12. □

We can do better still if we multiply the classes $\tau_1(i)$ and $\tau_{p^j}(k)$ for some integers $i, k \geq 0$.

Corollary A.14. *Let A be a division algebra with* $\mathrm{ind}(A) = p^n$ *and $r\mathcal{B}eh(A) = (n_0, ..., n_m)$. Let X be the Severi-Brauer variety of A. Pick an integer $0 \leq j \leq m$, and let $1 \leq i, k \leq p^n - 1$ be two integers with $i + k \leq p^n - 1$.*

There exists a number $\beta_{i,k}$ in $\mathbb{Z}_{(p)}$ so that $\beta_{i,k}\tau_1(i)\tau_{p^j}(k)$ is contained in $\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)}$. *Moreover, the p-adic valuation of the $\beta_{i,k}$ we find equals*

$$v_p(\beta_{i,k}) = \begin{cases} \max\{v_p(i)-j-n_j,0\} & \text{if} \quad 1\le k\le p^{n_j} \\ \max\{v_p(i)-j-n_j-\lfloor \log_p(k/p^{n_j})\rfloor,0\} & \text{if} \quad p^{n_j}<k\le p^{n-j} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as Corollary A.13 except that we use the equality, up to prime-to-p parts,

$$\pi^*(\beta_k\tau_1(i)\tau_{p^j}(k)) = p^{n+(k-1)j-v_p(k)+n-v_p(i)}h^{i+k}$$

to find $p^{v_p(\beta_{i,k})}\tau_1(i)\tau_{p^j}(k)$ is contained in N. □

As an application, the above can be used to settle the particular case of Conjecture 1.1 when X is the Severi-Brauer variety of an algebra A with level 1:

Theorem A.15. *Let A be a central simple k-algebra of level 1 and let X be the Severi-Brauer variety of A. Assume* $\mathrm{CH}(X)$ *is generated by Chern classes. Then the* K*-theory coniveau epimorphism* $\mathrm{CH}(X)\to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Proof. It's sufficient to show the claim when A is a division algebra of index p^n. In this case the kernel of the epimorphism $\mathrm{CH}(X)\to\mathrm{gr}_\tau\mathrm{G}(X)$ is p-primary-torsion so we can work with $\mathbb{Z}_{(p)}$ coefficients throughout the proof. Let L be a splitting field for A. Since $\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)}$ is p-torsion free, the composition

$$\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)}\to\mathrm{CH}(X)\otimes\mathbb{Z}_{(p)}\to\mathrm{gr}_\tau\mathrm{G}(X)\otimes\mathbb{Z}_{(p)}$$

is injective; we denote by C the image of this composition. We have an inequality

$$\text{(in)} \qquad [\mathrm{CH}(X)\otimes\mathbb{Z}_{(p)}:\mathrm{CT}(1;X)\otimes\mathbb{Z}_{(p)}]\ge[\mathrm{gr}_\tau\mathrm{G}(X)\otimes\mathbb{Z}_{(p)}:C].$$

We're going to use the bounds from Corollary A.14 to get an upper bound on the left of (in). We'll also bound the right of (in), by computing

$$[\mathrm{gr}_\tau\mathrm{G}(X)\otimes\mathbb{Z}_{(p)}:C]=\frac{[\mathrm{gr}_\tau\mathrm{G}(X_L):C]}{[\mathrm{K}(X_L):\mathrm{K}(X)]}$$

precisely; the equality of the ratio of these indices can be found in [Kar17a, proof of Theorem 3.1]. The proof will be completed once we show these two bounds are equal.

To get an upper bound on the left of (in), we sum the maximums of the p-adic valuations occurring in Corollaries A.13 and A.14. Plainly said, we compute an upper bound on p-adic valuations of the orders of the elements $\tau_1(i)\tau_{p^r}(k)$, where r is the (unique since A has level 1) smallest positive integer with $v_p(\mathrm{ind}(A^{\otimes p^r}))<v_p(\mathrm{ind}(A^{\otimes p^{r-1}}))-1$, in the group $\mathrm{CH}(X)/\mathrm{CT}(1;X)$. Note that, by Proposition A.5 and Proposition A.8 the elements $\tau_1(i)\tau_{p^r}(k)$ are exactly the generators of this quotient group so that by computing an upper bound on their orders and raising p to this upper bound, we also compute an upper bound on the index in the left of (in). Once we have this upper bound, we'll move on to give a lower bound for the right hand side of (in). These two bounds turn out to be equal, showing our upper bound on the orders were in fact their precise order.

Set $n_r = v_p(\mathrm{ind}(A^{\otimes p^r}))$ and $\ell = n - r - n_r$. When $i = 0$, we sum the contributions from Corollary A.13,

$$\sum_{a=1}^{p^{n_r}-1} n - r - n_r + \sum_{a=p^{n_r}}^{p^{n-r}-1} n - r - n_r - \lfloor \log_p(a/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)\ell + \sum_{b=0}^{\ell-1} \varphi(p^{n_r+b+1})(\ell - b)$$

where φ is the Euler totient function (we use this function to combine those terms a that have the same value of $\lfloor \log_p(a/p^{n_r}) \rfloor$; there are exactly $\varphi(p^{n_r+b+1}) = p^{n_r+b+1} - p^{n_r+b}$ such terms with value b, i.e. p^{n_r+b},..., $p^{n_r+b+1} - 1$). When $i > 0$, we only need to account for the terms with $v_p(i) > n - \ell$, (note if $\ell = 1$ then $r + n_r = n - 1$ and there are no terms of this kind),

$$\sum_{b=1}^{p^{n_r}-1} v_p(i) - r - n_r + \sum_{b=p^{n_r}}^{p^{v_p(i)-r}-1} v_p(i) - r - n_r - \lfloor \log_p(b/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)(v_p(i) - r - n_r) + \sum_{b=0}^{v_p(i)-r-n_r-1} \varphi(p^{n_r+b+1})(v_p(i) - r - n_r - b).$$

Of the integers i satisfying $1 \le i < p^n$ there are $\varphi(p^{\ell-1})$ integers i with $v_p(i) = n - \ell + 1$, there are $\varphi(p^{\ell-2})$ integers i with $v_p(i) = n - \ell + 2$, and so on to $\varphi(p)$ integers i with $v_p(i) = n - \ell + (\ell - 1)$. Summing over all such i with $v_p(i) > n - \ell$ we get

$$\sum_{a=1}^{\ell-1} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

Combining both the $i = 0$ and $i > 0$ contributions gives a definitive upper bound of

$$S = \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

To get a lower bound on the right of (in), we calculate $[\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C]$ precisely. Since this index equals

$$\frac{[\mathrm{gr}_\tau \mathrm{G}(X_L) : C]}{[\mathrm{K}(X_L) : \mathrm{K}(X)]},$$

it's sufficient to calculate the numerator and denominator of this fraction. The numerator depends only on the dimension of X and equals

$$\prod_{i=1}^{p^n} (p^{n-v_p(i)}) = \prod_{j=1}^{n-1} (p^{n-j})^{\varphi(p^{n-j})}.$$

The denominator depends on the reduced behavior of A and equals

$$\prod_{i=0}^{p^n-1} \mathrm{ind}(A^{\otimes i}) = \left(\prod_{j=0}^{r-1} (p^{n-j})^{\varphi(p^{n-j})} \right) \left(\prod_{j=r}^{n_r+r} (p^{n_r+r-j})^{\varphi(p^{n-j})} \right)$$

Dividing the two gives

$$P = \left(\prod_{i=r}^{n_r+r} (p^{\ell})^{\varphi(p^{n-i})} \right) \left(\prod_{i=n_r+r+1}^{n} (p^{n-i})^{\varphi(p^{n-i})} \right).$$

What remains to be shown is the equality $\log_p(P) = S$. A computation of the logarithm gives

$$\begin{aligned}\log_p(P) &= \log_p\left(\prod_{i=r}^{n_r+r}(p^\ell)^{\varphi(p^{n-i})}\prod_{i=n_r+r+1}^{n}(p^{n-i})^{\varphi(p^{n-i})}\right)\\ &= \sum_{i=r}^{n_r+r}\ell\varphi(p^{n-i}) + \sum_{i=n_r+r+1}^{n}(n-i)\varphi(p^{n-i})\\ &= \ell(p^{n-r}-p^{\ell-1}) + \sum_{i=1}^{n-r-n_r-1} i\varphi(p^i)\\ &= \ell(p^{n-r}-p^{\ell-1}) + \frac{(\ell-1)p^\ell - \ell p^{\ell-1}+1}{p-1}\\ &= \ell p^{n-r} - \frac{p^\ell-1}{p-1}.\end{aligned}$$

And by simplifying the sum S we find

$$\begin{aligned}S &= \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\left((p^{n_r}-1)a + \sum_{b=0}^{a}\varphi(p^{n_r+b+1})(a-b)\right)\\ &= \sum_{a=1}^{\ell}\varphi(p^{\ell-a})(p^{n_r}-1)a + \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\sum_{b=0}^{a}\varphi(p^{n_r+b+1})(a-b)\\ &= \frac{p^{n-r}-p^{n_r}}{p-1} - \frac{p^\ell-1}{p-1} + \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\left(\frac{p^{n_r}(p^{a+1}-(a+1)p+a)}{(p-1)}\right)\\ &= \frac{p^{n-r}-p^{n_r}}{p-1} - \frac{p^\ell-1}{p-1} + \frac{\ell p^{n-r+1}-(\ell+1)p^{n-r}+p^{n_r}}{p-1}\\ &= \ell p^{n-r} - \frac{p^\ell-1}{p-1}\end{aligned}$$

as desired. □

Appendix B. p-adic valuations

Fix a prime p to be used throughout this appendix. For any integer $n \geq 0$ we use $S_p(n)$ to denote the sum of the base-p digits of n. In other words, if $n = a_0 + a_1 p + \cdots + a_r p^r$ with $0 \leq a_0, ..., a_r \leq p-1$ then $S_p(n) = a_0 + a_1 + \cdots + a_r$. This appendix proves some simple results on the function S_p and on p-adic valuations involving this function.

Lemma B.1. *Let $n \geq 0$ be an integer.*

(1) $S_p(p^n) = 1$
(2) $S_p(p^n a) = S_p(a)$ *for any integer* $a \geq 0$
(3) $S_p(p^n - 1) = n(p-1)$
(4) *If* $0 \leq k \leq n$ *then* $S_p(p^n - p^k) = (n-k)(p-1)$
(5) *If* $0 \leq a \leq p^n$ *then* $S_p(p^n - a) + S_p(a) = (n - v_p(a))(p-1) + 1$
(6) *If* $0 \leq a \leq p^n - 1$ *then* $S_p(p^n - 1 - a) + S_p(a) = n(p-1)$

Proof. The proofs for (1)-(6) are elementary and omitted. □

We use the notation

$$\binom{n}{a_0, ..., a_r} = \frac{n!}{a_0! \cdots a_r!}.$$

If $a_0 + \cdots + a_r = n$ then we have the following:

Lemma B.2. *Let $n = a_0 + \cdots + a_r$ with $n, a_0, ..., a_r \geq 0$. Then*

$$v_p\left(\binom{n}{a_0, ..., a_r}\right) = \frac{1}{p-1}\left(\left(\sum_{i=0}^{r} S_p(a_i)\right) - S_p(n)\right).$$

Proof. See for example [Mer03, Lemma 11.2]. ☐

Lemma B.3. *Let $n > 0$ be an integer. Let $a_0, ..., a_r \geq 0$ be integers with $a_0 + \cdots + a_r = n$. Then*

$$v_p\left(\binom{n}{a_0, ..., a_r}\right) \geq v_p(n) - \min_{0 \leq i \leq r}\{v_p(a_i)\}.$$

Proof. See for example [Mer03, Lemma 11.3]. ☐

Lemma B.4. *Let $0 \leq r \leq s$ be integers. Fix an integer $0 < j \leq p^s$. Let $a_0, ..., a_{p^r} \geq 0$ be integers with $a_0 + \cdots + a_{p^r} = p^{s-r}$ and $a_1 + 2a_2 + \cdots + p^r a_{p^r} = j$. Write $j = s_0 p^r + s_1$ with $0 \leq s_1 < p^r$. Then if $s_1 = 0$ there is an inequality*

$$s - r - \min_{0 \leq k \leq p^r}\{v_p(a_k)\} + \sum_{i=1}^{p^r}(r - v_p(i))a_i \geq s - r - v_p(s_0)$$

and if $s_1 > 0$ there is an inequality

$$s - r - \min_{0 \leq k \leq p^r}\{v_p(a_k)\} - (r - v_p(s_1)) + \sum_{i=1}^{p^r}(r - v_p(i))a_i \geq s - r.$$

If $s_1 = 0$, then equality holds if and only if $a_0 = p^{s-r} - s_0$ and $a_{p^r} = s_0$. If $s_1 > 0$, then equality holds if and only if $a_0 = p^{s-r} - s_0 - 1$, $a_{s_1} = 1$, and $a_{p^r} = s_0$.

Proof. We first assume $s_1 = 0$. If $\ell = \min\{v_p(a_k)\}$ is 0, then the inequality clearly holds since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$. If $\ell > 0$ and $r = 0$, then $j = a_1$ and $j = s_0$. So ℓ is either $v_p(a_0) = v_p(p^s - j)$ or $v_p(a_1) = v_p(j) = v_p(s_0)$. Since $j \leq p^s$, it follows $\ell = v_p(s_0)$ and the claim follows with equality in this case. If $\ell = \min\{v_p(a_k)\} > 0$, then since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$, the inequality also holds if $r \neq 0$ and if there is a nonzero a_i with $i \neq 0, p^r$ as $(r - v_p(i))a_i - \ell \geq 0$.

Thus, to prove that the inequality holds in general (for $s_1 = 0$), it suffices to assume $\ell > 0$, $r > 0$, and $a_i = 0$ unless $i = 0$ or $i = p^r$. Assuming this is the case, it follows from the assumption $p^r a_{p^r} = j$ that $a_{p^r} = s_0$ and from the assumption $a_0 + a_{p^r} = p^{s-r}$ that $a_0 = p^{s-r} - s_0$. Since $s_0 \leq p^{s-r}$, we also have $v_p(a_{p^r}) \leq s - r$ so that $v_p(a_0) = v_p(a_{p^r})$ unless $a_{p^r} = p^{s-r}$ (in which case $v_p(a_0) = \infty$ and the claim is clear). Thus $\ell = v_p(s_0)$, the inequality holds, and it is even an equality in this case.

To see $a_0 = p^{s-r} - s_0$ and $a_{p^r} = s_0$ is the only case the inequality is an equality, one can work through the same cases. If $\ell = 0$ and there is equality, then $v_p(s_0) = 0$ and the large summation must equal 0. Hence $p^r a_{p^r} = j$ and the claim follows. If $\ell > 0$, then either $r = 0$ or $r > 0$. If $r = 0$, the claim follows from the first paragraph. If $r > 0$, then either all a_i with $i \neq 0, p^r$ vanish or there is at least one $0 < i < p^r$ with $a_i \neq 0$. We can assume the latter case where the inequality is a strict inequality since $(r - v_p(i))a_i - \ell \geq a_i - \ell > 0$.

To show the claim when $s_1 > 0$, we work through cases similar to before. Note now $r > 0$ holds always, as otherise we'd have $s_1 = 0$. If $\ell = \min\{v_p(a_k)\} = 0$ then since $r - v_p(i) \geq 0$, we're left to show the summation

$$\sum_{i=1}^{p^r}(r - v_p(i))a_i$$

is greater or equal $r - v_p(s_1) \leq r$. Since $s_1 > 0$, there is a smallest integer k with $0 \leq k \leq r-1$, $a_{bp^k} \neq 0$, and b relatively prime to p. It follows that p^k divides s_1 and $-(r - v_p(s_1)) \geq -r + k$. Since $(r - v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k)$ we find that the inequality holds by summing $(r - v_p(bp^k))a_{bp^k} - (r - v_p(s_1)) \geq (r-k) - (r-k) = 0$.

Thus to prove the inequality holds in general, it suffices to assume $\ell > 0$. Under our assumptions $\ell > 0$, $r > 0$, and $j \neq p^r a_{p^r}$ we have that there exists at least one i with $i \neq 0, p^r$ such that $a_i \neq 0$. Let k be the smallest integer between $0 \leq k < r$ such that $a_{bp^k} \neq 0$ for some b relatively prime to p. It follows p^k divides s_1 hence $-(r - v_p(s_1)) \geq -r + k$. Now

$$\begin{aligned}(r - v_p(bp^k))a_{bp^k} - r + v_p(s_1) - \ell &\geq (r-k)p^\ell - r + v_p(s_1) - \ell \\ &= (r-k)(p^\ell - 1) - \ell + v_p(s_1) \\ &\geq (p^\ell - 1 - \ell) + v_p(s_1) \\ &\geq 0.\end{aligned}$$

We end by showing that equality holds, assuming $s_1 > 0$, only in the specified case (it's clear equality holds in this case). We first assume $\ell = 0$. For equality to hold, we must have

$$\sum_{i=1}^{p^r}(r - v_p(i))a_i = r - v_p(s_1).$$

Again there is a minimal $0 \leq k < r$ with $a_{bp^k} \neq 0$ for some b relatively prime to p. We also get that p^k divides s_1. It follows

$$(r - v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k) \geq r - v_p(s_1)$$

must be an equality. Hence $a_{bp^k} = 1$ and we are in the specified case.

We next assume $\ell > 0$ and show our inequality is strict. Let k with $0 \leq k < r$ be minimal with $a_{bp^k} \neq 0$ for some b relatively prime to p. Then

$$\sum_{i=1}^{p^r}(r - v_p(i))a_i \geq (r-k)p^\ell.$$

Since $\ell + r - v_p(s_1) \leq \ell + r - k$ it suffices to check $(r-k)p^\ell > \ell + r - k$ holds for all $(r-k), \ell > 0$ in order to show this is a strict inequality in this case. But this is true since dividing by $r-k$ yields $p^\ell > \ell/(r-k) + 1$; making another estimate we can show $p^\ell > \ell + 1$ for all ℓ and this is always true for $\ell > 0$ and $p \geq 2$. □

Lemma B.5. *Let $0 \leq r \leq s$ be integers. Fix an integer $1 \leq j \leq p^s$ and write $j = s_0 p^r + s_1$ with $0 \leq s_1 < p^r$.*

If $s_1 = 0$, let $I = (a_0, ..., a_{p^r})$ be the tuple with $a_0 = p^{s-r} - s_0$, $a_{p^r} = s_0$ and $a_i = 0$ for all other i. Then,

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0) + S_p(a_{p^r}) - S_p(p^{s-r})) = s - r - v_p(s_0).$$

If $s_1 > 0$, let $I = (a_0, ..., a_{p^r})$ be the tuple with $a_0 = p^{s-r} - s_0 - 1$, $a_{s_1} = 1$, $a_{p^r} = s_0$ and $a_i = 0$ for all other i. Then,

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0) + S_p(a_{s_1}) + S_p(a_{p^r}) - S_p(p^{s-r})) = s - r.$$

Proof. The first equality follows from Lemma B.2 and Lemma B.1 (1) and (5). The second equality follows from Lemma B.2 and Lemma B.1 (1) and (6). ☐

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