ON THE ALGEBRAIZABILITY OF FORMAL DEFORMATIONS IN K-COHOMOLOGY

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Abstract. We study the higher direct image functors $R^i\pi_*\mathcal{F}$, associated to any S-scheme $\pi : X \to S$ and any abelian sheaf $\mathcal{F}$ on the big Zariski site of X, as sheaves on the big Zariski site of S. We note that these functors and their sheafifications, on the big étale and fppf sites over S, reflect the property of being locally of finite presentation when $\pi : X \to S$ is quasi-compact and quasi-separated. Our primary example, and interest in the remainder of this article, are the functors $R^i\pi_*\mathcal{K}_{n,X}$ associated to either Milnor or Quillen/Thomason K-theory sheaves.

We show that the functors $R^i\pi_*\mathcal{K}_{n,X}$ associated to the Milnor K-theory sheaves for $n = 2, 3$, and their étale sheafifications, have universal formal deformations when S is a scheme of finite type over an algebraic field extension $k/\mathbb{Q}$ and when $\pi : X \to S$ has smooth, proper, and geometrically connected fibers with some vanishing Hodge numbers. When $S = k$ is itself such a field, and if $i = n = 2$, then we show that the algebraizability of these universal formal deformations is a stable birational invariant of the scheme X.

Contents

1. Introduction

The problem of viewing the Chow groups of cycles on a given variety as representable by a group scheme is very old, very subtle, and very interesting. The first interesting case that was studied in great detail

Date: December 7, 2023.
2020 *Mathematics Subject Classification.* 14C35.

was the representability of the Chow group of codimension 1-cycles or, what amounts to the same, the study of the Picard scheme of a variety. Older approaches to representability, including Grothendieck's original construction of the Picard scheme, realized the Picard scheme as a quotient of an open subscheme of a component of the Hilbert scheme, see [Kle05] for a very detailed survey of the relevant facts.

Attempts to generalize these constructions to Chow groups of cycles of higher codimension were largely stopped with the results of [Mum68] where it was shown that the Chow group of codimension 2-cycles on an arbitrary surface could not, in general, be described as the group of rational points of an abelian variety in any natural way. Still, there were deformation theoretic results, such as those obtained by Bloch [Blo75], which showed that representability may still be achievable under some assumptions on the given variety (in this case, the assumptions were primarily the vanishing of certain Hodge numbers).

After the introduction of Bloch's results on the deformation theory of Chow groups of codimension 2-cycles, there was renewed interest in analyzing the representability of Chow groups of higher codimension cycles through the use of higher K-cohomology groups. Some of these attempts involved trying to create natural transformations from or to certain Hilbert schemes, see for example the work [Gra79] of Grayson. This avenue of research, however, seems to have mostly stalled due to the difficulty in constructing such natural transformations in general (although, there has been some recent development and interest in formal results in this direction, see [Yan18] or [DHY18]).

Recently, Benoist and Wittenberg [BW19] have constructed a functor using the K-theory of schemes (and, particularly, the graded object associated to the gamma filtration) in order to prove representability of the Chow group of codimension 2-cycles on a geometrically rational smooth and proper threefold. Their method is interesting but, it would be desirable to have a more flexible theory relying on a functor that can be defined for a wider class of varieties (see their comments in [BW19, Remarks 3.2] on the interest and difficulty in finding such a functor).

Surprisingly, at least to the author of the present article, there seems to have been no attempt in the literature to analyze the representability of higher K-cohomology groups with the use of Artin's representability criterion from [Art69]. The purpose of this article is to do exactly this, in an attempt to find a more natural setting in which to study the representability of Chow groups of higher codimension.

In this regard, we've had some success advancing the formal theory: in Section 2 and Section 4 we give a detailed construction of higher direct image functors $R^i\pi_*\mathcal{K}_{n,X}$, and their sheafifications in the étale topology, associated to a relative S-scheme $\pi : X \to S$ and to various different K-theory sheaves on the big site of X; in Section 3 we then show that the functors $R^i\pi_*\mathcal{K}_{n,X}$ are locally of finite presentation and this allows us, when combined with some observations on effective pro-representability in Section 5, to give new examples where the functors $R^2\pi_*\mathcal{K}_{2,X}$ and their étale sheafifications have algebraizable universal formal deformations (see Example 5.17).

One implication of our work, the idea behind which already appears in the work of Benoist and Wittenberg, is that the representability of codimension 2-cycles on a variety X should be determined by the stable birational equivalence class of X. As evidence for this claim, we show in Theorem 5.15 and Corollary 5.16 that for any variety $\pi : X \to k$ defined over an algebraic field extension $k/\mathbb{Q}$, the pro-representability, effective pro-representability, and algebraizability of the functor $R^2\pi_*\mathcal{K}_{2,X}$ is a stable birational invariant of X.

We try to work in a sufficient generality throughout the present text. For example, we work whenever possible over an arbitrary base scheme S and, although most of our contributions appearing in Section 5 use only Milnor K-theory sheaves, we also make it a point to include both Quillen K-theory and Thomason K-theory sheaves in our discussion in Section 4. We find it especially important to include the latter since it seems likely that natural transformations between higher direct image functors associated to K-theory sheaves will more easily be constructed using Quillen or Thomason K-theory than Milnor K-theory.

Acknowledgments. I'd like to thank Niranjan Ramachandran for both bringing my attention to the problem of representability of higher K-cohomology groups and for the many conversations that we've had about this work during its genesis.

2. Higher direct images

Let S be a fixed base and let $\pi : X \to S$ be an arbitrary S-scheme. In this section we recall the explicit description of the higher direct image functors $R^i\pi_*\mathcal{F}$ on the big Zariski site Sch/S associated to any abelian sheaf $\mathcal{F}$ on the big Zariski site Sch/X. By construction, these functors are the Zariski sheaves associated to the functor of S-schemes

$$T/S \rightsquigarrow \mathrm{H}^i(X \times_S T, \mathcal{F}_{X_T})$$

which have appeared frequently in the literature when the sheaf $\mathcal{F}$ is representable by a smooth group scheme, see e.g. [Oor62, AM77, BO21].

Note that, by [Sta21, Tag 0213], any abelian sheaf $\mathcal{F}$ on the big Zariski site Sch/S of a scheme S is determined by the following data

(SC1) for every scheme T/S, there is a specified abelian sheaf $\mathcal{F}_T$ on the underlying topological space of T,

(SC2) for every morphism of S-schemes $f : T' \to T$, there is a chosen morphism $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$,

which is, additionally, subject to the relations

(SR1) given two morphisms of S-schemes $f : T' \to T$ and $g : T'' \to T$, there is an equality $c_g \circ g^{-1}c_f = c_{f\circ g}$,

(SR2) if $f : T' \to T$ is an open immersion, then c_f is an isomorphism.

So, assume that $\mathcal{F}$ is an abelian sheaf on the big Zariski site Sch/X. To define higher direct image functors $R^i\pi_*\mathcal{F}$ on the big Zariski site Sch/S of S, we start by specifying the conditions **(SC1)** and **(SC2)**. We check after that **(SR1)** and **(SR2)** hold as needed.

2.1. **Objects.** Let T/S be an arbitrary S-scheme and let $\pi_T : X_T \to T$ be the map associated to T by base change. For any such T/S, we set

$$\left(R^i\pi_*\mathcal{F}\right)_T := R^i\pi_{T*}\mathcal{F}_{X_T}, \tag{1}$$

where $R^i\pi_{T*}\mathcal{F}_{X_T}$ is the sheaf on T associated to the presheaf assigning to an open $U \subset T$ the cohomology group $\mathrm{H}^i(\pi_T^{-1}(U), \mathcal{F}_{X_T}|_{\pi_T^{-1}(U)})$.

2.2. **Morphisms.** Let $\rho : T' \to T$ be any morphism of two S-schemes. To give an associated restriction map

$$c_\rho : \rho^{-1}R^i\pi_{T*}\mathcal{F}_{X_T} \to R^i\pi_{T'*}\mathcal{F}_{X_{T'}} \tag{2}$$

is equivalent, by adjunction, to giving a map

$$c_\rho^a : R^i\pi_{T*}\mathcal{F}_{X_T} \to \rho_*R^i\pi_{T'*}\mathcal{F}_{X_{T'}}. \tag{3}$$

We define c_ρ^a via base change from the following commutative diagram.

$$\begin{array}{ccc} X_{T'} & \xrightarrow{\rho_X} & X_T \\ \downarrow{\scriptstyle \pi_{T'}} & & \downarrow{\scriptstyle \pi_T} \\ T' & \xrightarrow{\rho} & T \end{array} \tag{4}$$

Specifically, note that there is a canonical morphism

$$\mathcal{F}_{X_T} \to \rho_{X*}\mathcal{F}_{X_{T'}} \tag{5}$$

which, composing with higher direct images, yields a morphism

$$R^i\pi_{T*}\mathcal{F}_{X_T} \xrightarrow{\phi_0} R^i\pi_{T*}\rho_{X*}\mathcal{F}_{X_{T'}}.$$

To get (3) we compose ϕ_0 with the composition

$$\begin{array}{ccc} R^i\pi_{T*}\rho_{X*}\mathcal{F}_{X_{T'}} \xrightarrow{\phi_1} R^i\left(\pi_T\circ\rho_X\right)_*\mathcal{F}_{X_{T'}} & & \\ \| & & \\ R^i(\rho\circ\pi_{T'})_*\mathcal{F}_{X_{T'}} & \xrightarrow{\phi_2} & \rho_*R^i\pi_{T'*}\mathcal{F}_{X_{T'}} \end{array}$$

where the first and last arrows are edge maps in the relative Leray spectral sequences for the two possible compositions $X_{T'}\to T$ in (4).

More precisely, the morphisms ϕ_i above, for $i=0,1,2$, can be defined as follows. We consider presheaves $\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2$ on T assigning to an open $U\subset T$ the cohomology groups

- $\mathcal{F}_0(U)=\mathrm{H}^i(\pi_T^{-1}(U),\mathcal{F}_{X_T}|_{\pi_T^{-1}(U)})$
- $\mathcal{F}_1(U)=\mathrm{H}^i(\pi_T^{-1}(U),(\rho_{X*}\mathcal{F}_{X_{T'}})|_{\pi_T^{-1}(U)})$
- $\mathcal{F}_2(U)=\mathrm{H}^i((\pi_T\circ\rho_X)^{-1}(U),\mathcal{F}_{X_{T'}}|_{(\pi_T\circ\rho_X)^{-1}(U)})$

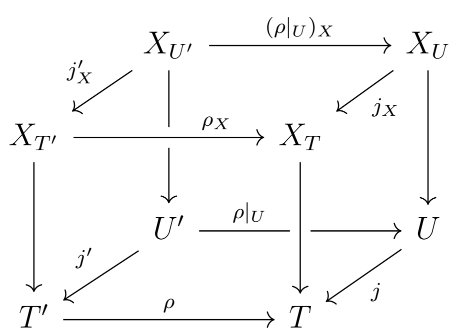
and a presheaf $\mathcal{F}_3$ on T' assigning to an open $U'\subset T'$ the cohomology

- $\mathcal{F}_3(U')=\mathrm{H}^i(\pi_{T'}^{-1}(U'),\mathcal{F}_{X_{T'}}|_{\pi_{T'}^{-1}(U')})$.

Writing $(-)^{\#}$ for the sheafification of a presheaf, we have

$$\mathcal{F}_0^{\#}=R^i\pi_{T*}\mathcal{F}_{X_T},\ \mathcal{F}_1^{\#}=R^i\pi_{T*}(\rho_{X*}\mathcal{F}_{X_{T'}}),\ \mathcal{F}_2^{\#}=R^i(\pi_T\circ\rho_X)_*\mathcal{F}_{X_{T'}}$$

and $\mathcal{F}_3^{\#}=R^i\pi_{T'*}\mathcal{F}_{X_{T'}}$. We will use the following commutative diagram and the notation introduced in it.



The canonical morphism of (5) is compatible with the above maps in the sense that there is a commuting diagram

$$\begin{array}{ccc} j_X^{-1}\mathcal{F}_{X_T} & \longrightarrow & j_X^{-1}\rho_{X*}\mathcal{F}_{X_{T'}} \\ \| & & \downarrow \\ \mathcal{F}_{X_U} & \longrightarrow & (\rho|_U)_{X*}\mathcal{F}_{X_{U'}} \end{array}$$

with the right vertical arrow an isomorphism when $U' = U \times_T T'$. To get the map ϕ_0, we consider an injective resolution $\mathcal{I}_\bullet$ of $(\rho|_U)_{X*}\mathcal{F}_{X_{U'}}$ and an injective resolution $\mathcal{J}_\bullet$ of $\mathcal{F}_{X_U}$ on X_U. Then there is a morphism of complexes filling in the dotted arrow below, unique up to homotopy,

$$\begin{array}{ccc} \mathcal{F}_{X_U} & \longrightarrow & \mathcal{J}_\bullet \\ \downarrow & & \vdots \\ (\rho|_U)_{X*}\mathcal{F}_{X_{U'}} & \longrightarrow & \mathcal{I}_\bullet \end{array}$$

which induces a unique morphism of cohomology

$$\mathrm{H}^i(\pi_T^{-1}(U), \mathcal{F}_{X_T}|_{X_U}) \to \mathrm{H}^i(\pi_T^{-1}(U), (\rho_{X*}\mathcal{F}_{X'_T})|_{\pi_T^{-1}(U)})$$

compatible with the restrictions of $\mathcal{F}_0$ and $\mathcal{F}_1$. The map ϕ_0 is then the associated map on sheafifications $\phi_0 : \mathcal{F}_0^\# \to \mathcal{F}_1^\#$.

To construct ϕ_1, we only need to construct morphisms on cohomology

$$\mathrm{H}^i(\pi_T^{-1}(U), (\rho_{X*}\mathcal{F}_{X_{T'}})|_{\pi_T^{-1}(U)}) \to \mathrm{H}^i(\pi_{T'}^{-1}(U'), \mathcal{F}_{X_{T'}}|_{\pi_{T'}^{-1}(U')}) \tag{6}$$

which are compatible with the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\mathcal{I}_\bullet$ be an injective resolution for $\mathcal{F}_{X_{U'}}$ on $X_{U'}$ and let $\mathcal{J}_\bullet$ be an injective resolution for $(\rho|_U)_{X*}\mathcal{F}_{X_{U'}}$ on X_U. Then there is a morphism of complexes filling in the dotted arrow below, unique up to homotopy,

$$\begin{array}{ccc} (\rho|_U)_X^{-1}(\rho|_U)_{X*}\mathcal{F}_{X_{U'}} & \longrightarrow & (\rho|_U)_X^{-1}\mathcal{J}_\bullet \\ \downarrow & & \vdots \\ \mathcal{F}_{X_{U'}} & \longrightarrow & \mathcal{I}_\bullet \end{array}$$

which gives a morphism $\mathcal{J}_\bullet \to (\rho|_U)_{X*}\mathcal{I}_\bullet$ by adjunction. Now the map induced on cohomology of this morphism of complexes yields a map of presheaves $\mathcal{F}_1 \to \mathcal{F}_2$ which sheafifies to ϕ_1.

Lastly, we construct ϕ_2. For this we construct a map $\mathcal{F}_2 \to \rho_*\mathcal{F}_3$. We get ϕ_2 through the universal property of sheafification and the canonical composition $\mathcal{F}_2 \to \rho_*\mathcal{F}_3 \to \rho_*(\mathcal{F}_3^\#)$. Noting that there's an equality $\pi_{T'}^{-1}(\rho^{-1}(U)) = (\pi_T \circ \rho_X)^{-1}(U)$ for every open $U \subset T$, the map $\mathcal{F}_2(U) \to \rho_*\mathcal{F}_3(U) = \mathcal{F}_3(\rho^{-1}(U))$ is just the identity.

2.3. **Relations.** The system $\{R^i\pi_{T*}\mathcal{F}_{X_T}, c_\rho\}_{T,\rho}$ given by (1) and (2) will act as the data of **(SC1)** and **(SC2)** for an abelian sheaf on the big Zariski site of S. To confirm that this data does define such a sheaf, we check the relations **(SR1)** and **(SR2)**.

Proposition 2.1. *Keep notation as above. Then the following is true for the morphisms defined as in (2):*

(1) if $f : T' \to T$ and $g : T'' \to T$ are two morphisms of S-schemes, then there is an equality $c_g \circ g^{-1}c_f = c_{f\circ g}$;
(2) if $f : T' \to T$ is an open immersion of S-schemes, then c_f is an isomorphism of sheaves on T'.

Proof. We first prove part (1). We use the Cartesian diagram below.

$$\begin{array}{ccccc} X_{T''} & \xrightarrow{g_X} & X_{T'} & \xrightarrow{f_X} & X_T \\ \downarrow{\scriptstyle \pi_{T''}} & & \downarrow{\scriptstyle \pi_{T'}} & & \downarrow{\scriptstyle \pi_T} \\ T'' & \xrightarrow{g} & T' & \xrightarrow{f} & T \end{array} \tag{7}$$

To see that the equality $c_g \circ g^{-1}c_f = c_{f\circ g}$ holds in general, it suffices to check that it holds on stalks; this will allow us to reduce the question of equality of these maps to a statement about functorality on the level of cohomology groups and, therefore, to the similar statement which holds for the sheaf $\mathcal{F}$ by assumption.

So let $t \in U \subset T$ be a point contained in an open subset U of T. We set $U' = f^{-1}(U)$ and $U'' = (f \circ g)^{-1}(U)$. The diagram (7) restricts over these open subsets to the Cartesian diagram below.

$$\begin{array}{ccccc} \pi_{T''}^{-1}(U'') & \xrightarrow{g_X} & \pi_{T'}^{-1}(U') & \xrightarrow{f_X} & \pi_T^{-1}(U) \\ \downarrow{\scriptstyle \pi_{T''}} & & \downarrow{\scriptstyle \pi_{T'}} & & \downarrow{\scriptstyle \pi_T} \\ U'' & \xrightarrow{g} & U' & \xrightarrow{f} & U \end{array} \tag{8}$$

Choose injective resolutions $\mathcal{I}''_\bullet$ of $\mathcal{F}_{X_{T''}}|_{\pi_{T''}^{-1}(U'')}$, and $\mathcal{I}'_\bullet$ of $\mathcal{F}_{X_{T'}}|_{\pi_{T'}^{-1}(U')}$, and $\mathcal{I}_\bullet$ of $\mathcal{F}_{X_T}|_{\pi_T^{-1}(U)}$. There are then maps of complexes coming from the adjunction of the conditions **(SC2)** for $\mathcal{F}$:

$$\begin{array}{ccc} \mathcal{F}_{X_T}|_{\pi_T^{-1}(U)} & \longrightarrow & \mathcal{I}_\bullet \\ \downarrow & & \downarrow \\ f_{X*}\mathcal{F}_{X'_T}|_{\pi_{T'}^{-1}(U')} & \longrightarrow & f_{X*}\mathcal{I}'_\bullet \\ \downarrow & & \downarrow \\ f_{X_*}g_{X*}\mathcal{F}_{X_{T''}}|_{\pi_{T''}^{-1}(U'')} & \longrightarrow & f_{X_*}g_{X*}\mathcal{I}''_\bullet. \end{array} \tag{9}$$

Applying the global sections functor yields the canonical map

$$\mathrm{H}^i(\pi_T^{-1}(U), \mathcal{F}_{X_T}|_{\pi_T^{-1}(U)}) \to \mathrm{H}^i(\pi_{T''}^{-1}(U''), \mathcal{F}_{X_{T''}}|_{\pi_{T''}^{-1}(U'')})$$

which determines the morphism of sheaves $c_g \circ g^{-1}c_f$ at the stalk level. The morphism $c_{f\circ g}$ similarly comes from a map on cohomology and, because of **(SR2)** for $\mathcal{F}$, the two are equal.

To prove part (2) it suffices to evaluate c_f on stalks and note that, if $f : T' \to T$ is an open immersion and $f_X : X_{T'} \to X_T$ the map gotten by base change, then $f_X^{-1}\mathcal{F}_{X_T} \cong \mathcal{F}_{X_{T'}}$ since $\mathcal{F}$ is a sheaf on the big Zariski site of X. We leave the details to the reader. □

2.4. **Higher direct images.** We can now give the definition for the higher direct image functors $R^i\pi_*\mathcal{F}$, of an abelian sheaf $\mathcal{F}$ on the big Zariski site Sch/X associated to an S-scheme $\pi : X \to S$, as an abelian sheaf on the big Zariski site Sch/S.

Definition 2.2. Let $\mathcal{F}$ be an abelian sheaf on the big Zariski site Sch/X. For any $i \geq 0$, we write $R^i\pi_*\mathcal{F}$ for the sheaf on the big Zariski site Sch/S associated, by [Sta21, Tag 0213], to the following data:

(HD1) for any scheme T/S, we take $(R^i\pi_*\mathcal{F})_T := R^i\pi_{T*}\mathcal{F}_{X_T}$ as in (1),

(HD2) and, for any morphism $\rho : T' \to T$ of S-schemes T and T', we take $c_\rho : \rho^{-1}R^i\pi_{T*}\mathcal{F}_{X_T} \to R^i\pi_{T'*}\mathcal{F}_{X_{T'}}$ to be the morphism defined as in (2).

That this data determines a well-defined abelian sheaf on the big Zariski site Sch/S follows from Proposition 2.1.

Remark 2.3. It's immediate to determine the explicit description of the functor $R^i\pi_*\mathcal{F} : \mathsf{Sch}/S \to \mathsf{Ab}$ on objects and morphisms of Sch/S from the above data. For any scheme T/S, one has

$$R^i\pi_*\mathcal{F}(T) = \mathrm{H}^0(T, R^i\pi_{T*}\mathcal{F}_{X_T})$$

and for any morphism $\rho : T' \to T$, the map $R^i\pi_*\mathcal{F}(T) \to R^i\pi_*\mathcal{F}(T')$ is the canonical map

$$\mathrm{H}^0(T, R^i\pi_{T*}\mathcal{F}_{X_T}) \to \mathrm{H}^0(T', R^i\pi_{T'*}\mathcal{F}_{X_{T'}})$$

gotten from the sheaf map c_ρ^a of (3).

Remark 2.4. Note that if $i = 0$, then $R^i\pi_*\mathcal{F} = \pi_*\mathcal{F}$.

Remark 2.5. In the following sections, we'll be interested in a special case of the above situation where one can say quite a bit more about the associated restriction maps for these higher direct image functors. In this remark, we suppose that $\pi : X \to k$ is a scheme over a fixed base field k, and we let $\mathcal{F}$ be an abelian sheaf on the big Zariski site Sch/X of X as before.

Let $T = \mathrm{Spec}(S)$ be the spectrum of a local k-algebra $(S, \mathfrak{m}_S)$ and let $\rho : T' \to T$ be an arbitrary morphism from a k-scheme T'. Then, since T is a local scheme, there is an isomorphism between the global sections functor, from the category of abelian sheaves on T to the category of abelian groups, and the functor taking the stalk of an abelian sheaf at the unique closed point corresponding to the maximal ideal $\mathfrak{m}_S \subset S$. Hence the associated map

$$R^i\pi_*\mathcal{F}(T) = \mathrm{H}^0(T, R^i\pi_{T*}\mathcal{F}_{X_T}) \to \mathrm{H}^0(T', R^i\pi_{T'*}\mathcal{F}_{X_{T'}}) = R^i\pi_*\mathcal{F}(T')$$

is isomorphic with the composition of the homomorphism,

$$\mathrm{H}^i(X_T, \mathcal{F}_{X_T}) \to \mathrm{H}^i(X_T, \rho_{X*}\mathcal{F}_{X_{T'}}), \tag{10}$$

induced by the morphism of sheaves in (5) above, followed by the first edge homomorphism

$$\mathrm{H}^i(X_T, \rho_{X*}\mathcal{F}_{X_{T'}}) \to \mathrm{H}^i(X_{T'}, \mathcal{F}_{X_{T'}}), \tag{11}$$

and, lastly, followed by the second edge homomorphism

$$\mathrm{H}^i(X_{T'}, \mathcal{F}_{X_{T'}}) \to \mathrm{H}^0(T', R^i\pi_{T'*}\mathcal{F}_{X_{T'}}). \tag{12}$$

If $T' = \mathrm{Spec}(R)$ is similarly the spectrum of a local k-algebra $(R, \mathfrak{m}_R)$, then the homomorphism (12) above is an isomorphism as well. In some cases, i.e. with some assumptions on S and R, it's also possible to show that the first edge homomorphism (11) in the above is an isomorphism.

Lemma 2.6. *Keep notation as above and assume that $\rho : T' \to T$ is the closed immersion associated to a surjective homomorphism $S \to R$. Then the edge homomorphism*

$$\mathrm{H}^i(X_T, \rho_{X*}\mathcal{F}_{X_{T'}}) \to \mathrm{H}^i(X_{T'}, \mathcal{F}_{X_{T'}})$$

from (11) is an isomorphism.

Proof. Under the assumptions of the lemma, the functor ρ_{X*} is known to be exact. The claim now follows from the explicit description of the edge homomorphism, see the paragraph following (6). □

The following lemma is not used in the rest of this text. However, we include it here as it seems that it could be useful in the future.

Lemma 2.7. *Suppose now that $(R, \mathfrak{m}_R)$ is an artinian local k-algebra with residue field $R/\mathfrak{m}_R \cong k$ and assume that $(S, \mathfrak{m}_S)$ is a local k-algebra with with $S/\mathfrak{m}_S \cong k$ as well.*

Let $\rho : T' \to T$ be a closed immersion associated to a surjective map $S \to R$ and write $\phi : T \to k$ for the structure map associated to the k-algebra map $k \to S$. We use the notation of the following commutative diagram gotten from base change to X:

$$\begin{array}{ccccc}
X_{T'} & \xrightarrow{\rho_X} & X_T & \xrightarrow{\phi_X} & X \\
\downarrow{\scriptstyle \pi_{T'}} & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\
T' & \xrightarrow{\rho} & T & \xrightarrow{\phi} & k.
\end{array}$$

Then, for any abelian sheaf $\mathcal{F}$ on $X_{T'}$, the canonical homomorphism

$$\mathrm{H}^i(X_{T'}, \mathcal{F}) = \mathrm{H}^i(X, \phi_{X*}\rho_{X*}\mathcal{F}) \to \mathrm{H}^i(X_T, \rho_{X*}\mathcal{F}) \tag{13}$$

is an isomorphism.

Proof. Note that the composition $\phi \circ \rho$ is a universal homeomorphism since the residue field of R is k ([Sta21, Tag 01S2, Tag 04DF]), so that the composition $\phi_X \circ \rho_X$ is a homeomorphism as well. In this way, we can canonically identify the groups $\mathrm{H}^i(X_{T'}, \mathcal{F})$ and $\mathrm{H}^i(X, \phi_{X*}\rho_{X*}\mathcal{F})$. The latter map in (13) is then the first edge homomorphism associated to the Leray spectral sequence for $\pi \circ \phi_X$ described in subsection 2.2.

To prove the claim, we'll argue that for any abelian sheaf $\mathcal{F}$ on $X_{T'}$, the higher direct images $R^i\phi_{X*}(\rho_{X*}\mathcal{F}) = 0$ (on the small site associated to the underlying topological space of X) vanish for every integer $i > 0$. The claim then follows from [Sta21, Tag 01F4].

Let $x \in X$ be any point. Then we can compute the stalk of the sheaf $R^i\phi_{X*}(\rho_{X*}\mathcal{F})$ at x as the colimit, over all opens $U \subset X$ containing x,

$$(R^i\phi_{X*}(\rho_{X*}\mathcal{F}))_x = \varinjlim_{x \in U} \mathrm{H}^i(\phi_X^{-1}(U), (\rho_{X*}\mathcal{F})|_{\phi_X^{-1}(U)}),$$

see [Har77, Chapter 3, Proposition 8.1]. Because x has a neighborhood basis consisting of affine open subsets of X, we can restrict the set of opens appearing in the colimit to only those opens which are affine. The restriction map

$$\mathrm{H}^i(\phi_X^{-1}(U), (\rho_{X*}\mathcal{F})|_{\phi_X^{-1}(U)}) \to \mathrm{H}^i(\phi_X^{-1}(W), (\rho_{X*}\mathcal{F})|_{\phi_X^{-1}(W)})$$

appearing in the above colimit, associated to the inclusion of affine opens $W \subset U$, fits into the following the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^i(\phi_X^{-1}(U), (\rho_{X*}\mathcal{F})|_{\phi_X^{-1}(U)}) & \longrightarrow & \mathrm{H}^i(\phi_X^{-1}(W), (\rho_{X*}\mathcal{F})|_{\phi_X^{-1}(W)}) \\
\| & & \| \\
\mathrm{H}^i(\phi_X^{-1}(U), (\rho_X|_{U_{T'}})_*(\mathcal{F}|_{U_{T'}})) & \longrightarrow & \mathrm{H}^i(\phi_X^{-1}(W), (\rho_X|_{W_{T'}})_*(\mathcal{F}|_{W_{T'}})) \\
\downarrow & & \downarrow \\
\mathrm{H}^i(U_{T'}, \mathcal{F}|_{U_{T'}}) & \longrightarrow & \mathrm{H}^i(W_{T'}, \mathcal{F}|_{W_{T'}}).
\end{array}$$

The vertical arrows in the top square are natural isomorphisms gotten by base change for the given sheaves [Sta21, Tag 0FN2]; the vertical arrows in the bottom square are the canonical edge homomorphisms which, by Lemma 2.6, are also isomorphisms.

Since W and U are affine, the maps $W_{T'} \to U_{T'}$ are spectral and it follows from both the above work and [Sta21, Tag 0A37] that there are isomorphisms

$$(R^i\phi_{X*}(\rho_{X*}\mathcal{F}))_x \cong \varinjlim_{x\in U} \mathrm{H}^i(U_{T'}, \mathcal{F}|_{U_{T'}}) \cong \mathrm{H}^i(Z, \mathcal{F}|_Z)$$

where $Z = \varprojlim_U U \times_k T' = \mathrm{Spec}(\mathcal{O}_{X,x} \otimes_k R)$, cf. [Sta21, Tag 01YZ]. Since R is assumed to be an artinian k-algebra with $R/\mathfrak{m}_R \cong k$, the ring $\mathcal{O}_{X,x} \otimes_k R$ is a local ring [Swe75]. But this implies that Z is a local scheme, so that $\mathrm{H}^i(Z, \mathcal{F}|_Z) = 0$ for all $i > 0$ as desired. □

2.5. **Sheafifications.** If $\pi : X \to S$ is any S-scheme, then for any abelian sheaf $\mathcal{F}$ on the big Zariski site Sch/X of X, and for any $i \geq 0$, we've constructed the higher direct image functor $R^i\pi_*\mathcal{F}$ as a sheaf on the big Zariski site Sch/S of S. If τ is any topology on Sch/S finer than the Zariski topology, then we get an associated sheafification $(R^i\pi_*\mathcal{F})_\tau$ of $R^i\pi_*\mathcal{F}$ which is a sheaf for the τ-topology.

In this article, the topologies on the category Sch/S that we will be primarily interested in are the fppf, étale, and Zariski topologies. Associated with the canonical comparsions of the given big sites for these topologies are natural comparison morphisms of functors

$$R^i\pi_*\mathcal{F} \to (R^i\pi_*\mathcal{F})_{\acute{e}t} \to (R^i\pi_*\mathcal{F})_{fppf}. \tag{14}$$

Remark 2.8. If S'/S is any extension of S, then there is a commutative square of morphisms of big Zariski sites

$$\begin{array}{ccc} \mathsf{Sch}/X_{S'} & \longrightarrow & \mathsf{Sch}/S' \\ \downarrow & & \downarrow \\ \mathsf{Sch}/X & \longrightarrow & \mathsf{Sch}/S. \end{array}$$

From this we get natural isomorphisms

$$R^i\pi_{S'*}(\mathcal{F}|_{X_{S'}}) \cong (R^i\pi_*\mathcal{F})|_{S'} \tag{15}$$

showing that formation of higher direct image functors is compatible with changing the base (cf. [Sta21, Tag 0EYV] in the case $i = 0$).

If τ is any topology on the category of schemes which is finer than the Zariski topology, then the same is true for the sheafifications of the higher direct image functors in the τ-topology. More precisely, there is a commutative square of morphisms of sites

$$\begin{array}{ccc} (\mathsf{Sch}/S')_\tau & \longrightarrow & \mathsf{Sch}/S' \\ \downarrow & & \downarrow \\ (\mathsf{Sch}/S)_\tau & \longrightarrow & \mathsf{Sch}/S. \end{array}$$

Restriction along either of the horizontal arrows takes a sheaf to its sheafification in the τ-topology [Sta21, Tag 0EWI], so

$$(R^i\pi_{S'*}(\mathcal{F}|_{X_{S'}}))_\tau \cong ((R^i\pi_*\mathcal{F})|_{S'})_\tau \cong ((R^i\pi_*\mathcal{F})_\tau)|_{S'} \tag{16}$$

by the functorality of restriction along the composition of morphisms of sites [Sta21, Tag 03CB]

3. Locally of finite presentation

In this section we fix a base scheme S and an S-scheme $\pi : X \to S$. Given any functor $F : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ and any inverse system $\{T_i\}_{i\in I}$ of schemes over S with inverse limit T, there is a natural map

$$\varinjlim_i F(T_i) \to F(T). \tag{17}$$

Recall that such a functor F is said to be *locally of finite presentation* if the map (17) is a bijection for every affine scheme T over S that can be written as an inverse limit $T = \varprojlim T_i$ of a filtered inverse system of affine schemes $\{T_i\}_{i\in I}$ over S.

Proposition 3.1. *Suppose that π is quasi-compact and quasi-separated. Let $\mathcal{F}$ be an abelian sheaf for the big Zariski site Sch/X and assume that $\mathcal{F}$ is locally of finite presentation. Then, for any integer $i \geq 0$, the higher direct image functors*

$$R^i\pi_*\mathcal{F}, \quad (R^i\pi_*\mathcal{F})_{\acute{e}t}, \quad \textit{and} \quad (R^i\pi_*\mathcal{F})_{fppf}$$

are all locally of finite presentation.

Proof of Proposition 3.1. We will show, under the assumptions of the proposition, that the ith cohomology functor on S-schemes

$$T/S \to \mathrm{H}^i(X \times_S T, \mathcal{F}_{X_T})$$

is locally of finite presentation. Then it follows directly from [Sta21, Tag 049O] that the fppf sheafification $(R^i\pi_*\mathcal{F})_{fppf}$ of this functor is also locally of finite presentation. To get the same result for the Zariski and étale sheafifications, one can observe that the proof in [Sta21, Tag 049O] goes through virtually without change in these cases also, so that $R^i\pi_*\mathcal{F}$ and $(R^i\pi_*\mathcal{F})_{\acute{e}t}$ are locally of finite presentation too.

So let $\{T_j = \mathrm{Spec}(A_j), \rho_{j',j} : T_{j'} \to T_j\}_{j\in J}$ be a filtered inverse system of affine schemes over S and let $T = \mathrm{Spec}(A)$, with $A = \varinjlim A_j$, be the inverse limit considered as a scheme over S. For any index $j \in J$, we write $\rho_j : T \to T_j$ for the projection map. We start by noting that the maps of **(SC2)**, associated to any of the arrows $\rho_{j',j} : T_{j'} \to T_j$, yield a directed system of sheaves on $X \times_S T$

$$\rho_{j,X}^{-1}\mathcal{F}_{X_{T_j}} \to \rho_{j',X}^{-1}\mathcal{F}_{X_{T_{j'}}}.$$

The colimit of this directed system admits a canonical morphism

$$\mathcal{L} := \varinjlim_j \rho_{j,X}^{-1}\mathcal{F}_{X_{T_j}} \to \mathcal{F}_{X_T} \tag{18}$$

and the comparison map $\varinjlim_j \mathrm{H}^i(X \times_S T_j, \mathcal{F}_{X_{T_j}}) \to \mathrm{H}^i(X \times_S T, \mathcal{F}_{X_T})$ factors via

$$\varinjlim_j \mathrm{H}^i(X \times_S T_j, \mathcal{F}_{X_{T_j}}) \xrightarrow{\sim} \mathrm{H}^i(X \times_S T, \mathcal{L}) \to \mathrm{H}^i(X \times_S T, \mathcal{F}_{X_T}).$$

Here the left arrow is the canonical isomorphism of [Sta21, Tag 0A37] and we are using the assumption that $\pi : X \to S$ is quasi-compact and quasi-separated in order to realize the topological space $X \times_S T$ as a spectral space written as the limit of spectral spaces $X \times_S T_j$ along quasi-compact, and hence spectral, maps, cf. [Sta21, Tag 08YF]. Therefore, to complete the proof, it suffices to show that the morphism of sheaves in (18) is an isomorphism.

This is a claim that can be checked on stalks and, for any $t \in X \times_S T$, there are canonical isomorphisms

$$\begin{aligned}
\left(\varinjlim_j \rho_{j,X}^{-1}\mathcal{F}_{X_{T_j}}\right)_t &\cong \varinjlim_j \left(\rho_{j,X}^{-1}\mathcal{F}_{X_{T_j}}\right)_t \\
&\cong \varinjlim_j \left(\mathcal{F}_{X_{T_j}}\right)_{\rho_{j,X}(t)} \\
&\cong \varinjlim_j \mathcal{F}(\mathrm{Spec}(\mathcal{O}_{X_{T_j},\rho_{j,X}(t)})) \\
&\cong \mathcal{F}(\varprojlim_j \mathrm{Spec}(\mathcal{O}_{X_{T_j},\rho_{j,X}(t)})) \\
&\cong \mathcal{F}(\mathrm{Spec}(\mathcal{O}_{X_T,t})) \\
&\cong (\mathcal{F}_{X_T})_t\,.
\end{aligned}$$

In the above we've used the locally finite presentation assumption on $\mathcal{F}$ to go from line 3 to line 4, by taking the inverse limit of the system of affine open neighborhoods containing $\rho_{j,X}(t)$ to identify $(\mathcal{F}_{X_{T_j}})_{\rho_{j,X}(t)}$ and $\mathcal{F}(\mathrm{Spec}(\mathcal{O}_{X_{T_j},\rho_{j,X}(t)}))$, as well as to go from line 4 to line 5. $\square$

As an immediate corollary to Proposition 3.1, we get a computation for the stalks of the Zariski and étale higher direct image functors:

Corollary 3.2. *Let $\pi : X \to S$ be quasi-compact and quasi-separated. Let T/S be an arbitrary scheme over S, pick a point $t \in T$, and choose a geometric point $\bar{t} : \mathrm{Spec}(\Omega) \to T$ lying over t (i.e. the ring Ω is an algebraically closed field and $\bar{t}$ corresponds to an embedding $\kappa(t) \subset \Omega$). Let $R = \mathrm{Spec}(\mathcal{O}_{T,t})$ be the spectrum of the local ring $\mathcal{O}_{T,t}$ of $t \in T$ and let $R^{sh} = \mathrm{Spec}(\mathcal{O}^{sh}_{T,t})$ be the spectrum of the strict henselization $\mathcal{O}^{sh}_{T,t}$. Then there are natural isomorphisms*

$$\left(R^i\pi_*\mathcal{F}\right)_t = R^i\pi_*\mathcal{F}(R) = \mathrm{H}^i(X_R, \mathcal{F}_{X_R}) \tag{19}$$

and

$$\left(R^i\pi_*\mathcal{F}\right)_{\text{ét},\bar{t}} = \left(R^i\pi_*\mathcal{F}\right)_{\text{ét}}(R^{sh}) = \mathrm{H}^i(X_{R^{sh}}, \mathcal{F}_{X_{R^{sh}}}) \tag{20}$$

for any abelian sheaf $\mathcal{F}$ on the big Zariski site Sch/X of X which is locally of finite presentation.

Proof. The isomorphism on the left in (19) is a direct consequence of Proposition 3.1. To see the right isomorphism in (19), we note that

$$R^i\pi_*\mathcal{F}(R) = \mathrm{H}^0(R, R^i\pi_{R*}\mathcal{F}_{X_R})$$

and the right hand side is canonically $\mathrm{H}^i(X_R, \mathcal{F}_{X_R})$ since R is local.

For the equation (20) with étale sheaves we recall that, by definition, the stalk at $\bar{t}$ of any étale presheaf is computed as the colimit

$$\left(R^i\pi_*\mathcal{F}\right)_{\text{ét},\bar{t}} = \varinjlim_{(U,\overline{u})} \left(R^i\pi_*\mathcal{F}\right)_{\text{ét}}(U),$$

where the index runs over all étale neighborhoods $(U, \overline{u})$ of $\bar{t}$. We can consider only affine schemes in the colimit without changing anything. The isomorphism on the left of equation (20) now follows directly from Proposition 3.1 and [Sta21, Tag 04GW].

By [Sta21, Tag 03PT], there is a canonical isomorphism

$$\left(R^i\pi_*\mathcal{F}\right)_{\text{ét},\bar{t}} = (R^i\pi_*\mathcal{F})_{\bar{t}}$$

where the right hand side is the stalk at $\bar{t}$ of the Zariski sheaf considered as an étale presheaf. As before, the right hand side here is $R^i\pi_*\mathcal{F}(R^{sh})$ which, since R^{sh} is local, equals $\mathrm{H}^i(X_{R^{sh}}, \mathcal{F}_{X_{R^{sh}}})$. □

Remark 3.3. Let $\pi : X \to k$ be any scheme over a fixed base field k. Let F/k be any field extension of k. Then there are isomorphisms

$$R^i\pi_*\mathcal{F}(F) = \mathrm{H}^0(F, R^i\pi_{F*}\mathcal{F}_{X_F}) = \mathrm{H}^i(X_F, \mathcal{F}_{X_F}).$$

If π is quasi-compact and quasi-separated, and if $\mathcal{F}$ is locally of finite presentation, then letting F^s denote a fixed separable closure of F with absolute Galois group $G_F = \mathrm{Gal}(F^s/F)$ we find (e.g. by applying

[Mil80, Chapter II, Proposition 1.4] at the finite level and writing F^s as a colimit) that

$$(R^i\pi_*\mathcal{F})_{\acute{e}t}(F) = \mathrm{H}^i(X_{F^s}, \mathcal{F}_{X_{F^s}})^{G_F}.$$

Similarly, if $(A, \mathfrak{m}_A)$ is a local artinian F-algebra with residue field $A/\mathfrak{m}_A \cong F$, then [Sta21, Tag 03SI] implies that

$$(R^i\pi_*\mathcal{F})_{\acute{e}t}(R') = \mathrm{H}^i(X_{R'_{F^s}}, \mathcal{F}_{X_{R'_{F^s}}})^{G_F}$$

where $R' = \mathrm{Spec}(A)$. If $(B, \mathfrak{m}_B)$ is a second local artinian F-algebra with $B/\mathfrak{m}_B \cong F$ which admits a surjection $B \to A$ of local F-algebras then, because of Remark 2.5 and Lemma 2.6, the induced map

$$(R^i\pi_*\mathcal{F})_{\acute{e}t}(R) \to (R^i\pi_*\mathcal{F})_{\acute{e}t}(R'),$$

where $R = \mathrm{Spec}(B)$, is equivalent to the map gotten by taking G_F-invariants of the canonical reduction map

$$\mathrm{H}^i(X_{R_{F^s}}, \mathcal{F}_{X_{R_{F^s}}}) \to \mathrm{H}^i(X_{R'_{F^s}}, \mathcal{F}_{X_{R'_{F^s}}})$$

induced by the sheaf morphism in (5).

4. K-theory sheaves

If $\pi : X \to S$ is any given S-scheme, then the primary examples of abelian sheaves on the big Zariski site Sch/X that we'll be interested in, in this text, come from constructions in K-theory. There are three such families of sheaves that we can define on Sch/X and each of them comes from restriction, along the unique morphism of sites $\mathsf{Sch}/X \to \mathsf{Sch}/\mathbb{Z}$, of families of sheaves defined on $\mathsf{Sch}/\mathbb{Z}$.

Specifically, for any integer $n \geq 0$ we'll consider the nth Milnor, Quillen, and Thomason K-theory sheaves on the big Zariski site $\mathsf{Sch}/\mathbb{Z}$; we denote these sheaves by

$$\mathcal{K}^M_{n,\mathbb{Z}}, \quad \mathcal{K}^Q_{n,\mathbb{Z}}, \quad \text{and} \quad \mathcal{K}^T_{n,\mathbb{Z}} \tag{21}$$

respectively. We briefly recall how each of these sheaves is constructed, in the subsections below, along with some of their natural compatibility.

4.1. **Milnor K-theory.** We handle the Milnor K-theory sheaves first. For a commutative ring R, and for any integer $n \geq 0$, we write $K^M_n(R)$ for the degree n graded summand of the graded quotient ring

$$K^M_*(R) = T_{\mathbb{Z}}(R^\times)/I,$$

where $T_{\mathbb{Z}}(R^\times)$ is the graded tensor $\mathbb{Z}$-algebra of the group of units $R^\times$ and $I \subset T_{\mathbb{Z}}(R^\times)$ is the homogeneous two-sided ideal generated by all of those elements of the form $a \otimes (1-a)$ with both a and $1-a$ in $R^\times$.

For a scheme Y, we let $\mathcal{K}^M_{n,Y}$ denote the abelian sheaf on the underlying topological space of Y associated to the presheaf

$$(U \overset{open}{\subset} Y) \rightsquigarrow K^M_n(\mathcal{O}_Y(U))$$

with restriction maps induced by the restriction maps of the structure sheaf $\mathcal{O}_Y$. For any $n \geq 0$, the sheaf $\mathcal{K}^M_{n,\mathbb{Z}}$ on the big Zariski site $\mathsf{Sch}/\mathbb{Z}$ is then that sheaf, specified by [Sta21, Tag 0213], with **(SC1)** given by

$$(\mathcal{K}^M_{n,\mathbb{Z}})_Y = \mathcal{K}^M_{n,Y}$$

for any scheme Y, and with **(SC2)** for a morphism $\rho : Y' \to Y$ given by adjunction of the canonical maps

$$\mathcal{K}^M_{n,Y} \to \rho_* \mathcal{K}^M_{n,Y'}$$

induced by the corresponding map of presheaves and the structural morphism $\mathcal{O}_Y \to \rho_* \mathcal{O}_{Y'}$. In this case the properties **(SR1)** and **(SR2)** follow from the functorality of Milnor K-theory.

4.2. **Quillen K-theory.** Next, we define the Quillen K-theory sheaves on the big site $\mathsf{Sch}/\mathbb{Z}$. Here there are two, *a priori* possibly different, constructions of such sheaves for any integer $n \geq 0$. However, it turns out that both constructions give the same result. Let Y be any scheme, let $n \geq 0$ be an integer, and consider the two presheaves $\mathcal{K}^{Q,r}_{n,Y}$ and $\mathcal{K}^{Q,s}_{n,Y}$ on the underlying topological space of Y defined by the associations

$$(U \overset{open}{\subset} Y) \rightsquigarrow K^Q_n(\mathcal{O}_Y(U)) \quad \text{and} \quad (U \overset{open}{\subset} Y) \rightsquigarrow K^Q_n(U)$$

respectively; here we are using the Quillen K-theory of rings for $\mathcal{K}^{Q,r}_{n,Y}$, i.e. the Quillen K-theory of the exact category of finitely generated projective modules, and of schemes for $\mathcal{K}^{Q,s}_{n,Y}$, i.e. the Quillen K-theory of locally free sheaves of finite rank, as defined in [Qui73].

Due to the equivalence between locally free sheaves of finite rank on an affine scheme and finitely generated projective modules under the corresponding ring, for any open $U \subset Y$ there is an identification

$$K^Q_n(\mathcal{O}_Y(U)) = K^Q_n(\mathrm{Spec}(\mathcal{O}_Y(U)))$$

which, along with the canonical map of schemes $U \to \mathrm{Spec}(\mathcal{O}_Y(U))$, produces a homomorphism

$$K^Q_n(\mathcal{O}_Y(U)) = K^Q_n(\mathrm{Spec}(\mathcal{O}_Y(U))) \to K^Q_n(U)$$

that extends to a natural transformation $\mathcal{K}^{Q,r}_{n,Y} \to \mathcal{K}^{Q,s}_{n,Y}$. The induced map on sheafifications is an isomorphism, which can be checked on the level of stalks by [Qui73, Proposition 2.2] and [Qui73, Example, p. 104]. We write $\mathcal{K}^Q_{n,Y}$ for either of these canonically identified sheaves on Y.

Now for any $n \geq 0$ we define the nth Quillen K-theory sheaf $\mathcal{K}^Q_{n,\mathbb{Z}}$ on the big Zariski site $\mathsf{Sch}/\mathbb{Z}$ by specifying, for the condition **(SC1)**,

$$(\mathcal{K}^Q_{n,\mathbb{Z}})_Y = \mathcal{K}^Q_{n,Y}$$

for any scheme Y. We also use, for the condition **(SC2)** for a morphism of schemes $\rho : Y' \to Y$, the morphism given by adjunction of the canonical morphism

$$\mathcal{K}^Q_{n,Y} \to \rho_* \mathcal{K}^Q_{n,Y'}$$

induced by the corresponding transformation of presheaves associated to the structure map $\mathcal{O}_Y \to \rho_* \mathcal{O}_{Y'}$. Conditions **(SR1)** and **(SR2)** then follow from the functorality of Quillen's K-theory.

4.3. **Thomason K-theory.** Lastly, we treat the Thomason K-theory sheaves. As in the case of both Milnor and Quillen K-theory, for any integer $n \geq 0$ the nth Thomason K-theory sheaf $\mathcal{K}^T_{n,\mathbb{Z}}$ on the big Zariski site $\mathsf{Sch}/\mathbb{Z}$ is gotten by specifying sheaves $\mathcal{K}^T_{n,Y}$, for any scheme Y, so that one can take

$$(\mathcal{K}^T_{n,\mathbb{Z}})_Y = \mathcal{K}^T_{n,Y}$$

for the condition **(SC1)** of $\mathcal{K}^T_{n,\mathbb{Z}}$; the condition **(SC2)** will also come naturally as the morphism given by adjunction of canonical maps

$$\mathcal{K}^T_{n,Y} \to \rho_* \mathcal{K}^T_{n,Y'}$$

induced by a morphism of schemes $\rho : Y' \to Y$. Lastly, the conditions **(SR1)** and **(SR2)** will both follow from the functorality of Thomason's K-theory.

We take, as the definition for the nth Thomason K-theory sheaf $\mathcal{K}^T_{n,Y}$ on the topological space underlying a scheme Y, the sheafification of the presheaf

$$(U \overset{open}{\subset} Y) \rightsquigarrow K^T_n(U)$$

which assigns to an open $U \subset Y$ the nth K-group $K^T_n(U)$ which, by definition, is the nth homotopy group of the K-theory spectrum of the complicial biWaldhausen category of perfect complexes of finite Tor-dimension in the abelian category of all chain complexes of $\mathcal{O}_U$-modules with cofibrations the degree-wise split monomorphisms and with weak equivalences being quasi-isomorphisms, see [TT90, Definition 3.1].

4.4. **Restriction.** For any integer $n \geq 0$ and for each of Milnor, Quillen, and Thomason K-theory, i.e. for any of $* = M, Q, T$, we write

$$\mathcal{K}^*_{n,X} := (\mathcal{K}^*_{n,\mathbb{Z}})|_X \tag{22}$$

for the restriction of the Zariski sheaf $\mathcal{K}^*_{n,\mathbb{Z}}$ to the big Zariski site Sch/X. If now Y/X is any X-scheme, then there is an equality $(\mathcal{K}^*_{n,X})_Y = \mathcal{K}^*_{n,Y}$.

Noticeably, there is some overlap in our notation between the K-theory sheaves on both the big and small Zariski sites of X but, this does not seem to cause any confusion.

4.5. **Locally of finite presentation.** For any S-scheme $\pi : X \to S$, and for any integer $n \geq 0$, each of the sheaves $\mathcal{K}^M_{n,X}$, $\mathcal{K}^Q_{n,X}$, and $\mathcal{K}^T_{n,X}$ is locally of finite presentation. This follows from the analogue of [Sta21, Tag 049O] for the Zariski topology, noting that each of the above three functors is (the restriction of) the sheafification of a presheaf that is locally of finite presentation (and also the restriction and sheafification operations commute, cf. Remark 2.8).

For Milnor K-theory, local finite presentation of the given presheaf is immediate from the definition; for Quillen K-theory this is due to [Qui73, Example, p. 104] or [Qui73, Section 7, Proposition 2.2] and for Thomason K-theory it follows from [TT90, Proposition 3.20].

4.6. **Compatibility.** For a fixed scheme X, and for any integer $n \geq 0$, there are natural transformations of sheaves on the big site Sch/X

$$\mathcal{K}^M_{n,X} \xrightarrow{\psi^Q_M(n)} \mathcal{K}^Q_{n,X} \xrightarrow{\psi^T_Q(n)} \mathcal{K}^T_{n,X} \tag{23}$$

and we summarize here what's known about them. The transformation $\psi^Q_M(n)$ is defined at the level of rings: for any commutative ring R, the direct sum of Quillen K-groups $K^Q_*(R) = \bigoplus_{n\geq 0} K^Q_n(R)$ has the structure of a graded commutative ring and there is a homomorphism

$$R^\times = K^M_1(R) \to K^Q_1(R);$$

this homomorphism is natural in R and is an isomorphism whenever R is local [Wei13, Chapter III, Lemma 1.4]. Multiplication in $K^Q_*(R)$ then induces a natural map from the tensor algebra $T_{\mathbb{Z}}(R^\times)$ to $K^Q_*(R)$ which, due to both [Wei13, Chapter III, Lemma 5.10] and [Wei13, Chapter IV, Example 1.10.1], induces group homomorphisms $K^M_n(R) \to K^Q_n(R)$. This yields natural transformations of presheaves for any $n \geq 0$

$$(U \overset{open}{\subset} Y) \rightsquigarrow \left(K^M_n(\mathcal{O}_Y(U)) \to K^Q_n(\mathcal{O}_Y(U))\right)$$

which sheafify to the transformations $\psi^Q_M(n)$.

The natural transformation $\psi^T_Q(n)$ is defined at the level of schemes: for any scheme Y, the K-theory spectrum $K^{naive}(Y)$ of the complicial biWaldhausen category of strictly perfect complexes in the category of all chain complexes of $\mathcal{O}_Y$-modules, with weak equivalences being quasi-isomorphisms and cofibrations degree-wise split monomorphisms, has the property $\pi_n(K^{naive}(Y)) = K^Q_n(Y)$ by [TT90, Proposition 3.10]. For any integer $n \geq 0$, the inclusion of the category of strictly perfect complexes of $\mathcal{O}_Y$-modules into the category of perfect complexes of $\mathcal{O}_Y$-modules with finite Tor-dimension induces a group homomorphism $K_n^Q(Y) \to K_n^T(Y)$ that's natural in Y. The natural transformations of presheaves that result from these maps

$$(U \overset{open}{\subset} Y) \rightsquigarrow \big(K_n^Q(U) \to K_n^T(U)\big)$$

sheafify to the transformations $\psi_Q^T(n)$.

For every integer $n \geq 0$, the comparison $\psi_Q^T(n)$ is an isomorphism. This can be checked on stalks since, for any scheme Y and any affine open subscheme $U \subset Y$, the map $K_n^Q(U) \to K_n^T(U)$ is an isomorphism [TT90, Corollary 3.9] since $\mathcal{O}_U$ itself is an ample family of line bundles. The transformations $\psi_M^Q(n)$ are also known to be isomorphisms if either $n \leq 1$ or $n = 2$ and X has infinite residue fields. This is clear for $n = 0$, it follows from [Wei13, Chapter III, Lemma 1.4] for $n = 1$, and it follows from [vdK77] in the case $n = 2$ (see also [NS89, Corollary 4.3]).

4.7. **Examples.** Let k be any field and fix a k-scheme $\pi : X \to k$. For some low values of the integers i and n, one can often idnetify the higher direct images $R^i\pi_*\mathcal{K}_{n,X}$ with other, more well-known, functors. We consider the cases where $n \leq i \leq 1$ below, and we write $R^i\pi_*\mathcal{K}_{n,X}$ for any of the higher direct images associated to any of the (canonically identified) K-theory sheaves on the big Zariski site of X.

To start, let $\underline{\mathbb{Z}}_X^b$ be the constant sheaf on the big Zariski site of X associated to the abstract group $\mathbb{Z}$. There is a natural transformation

$$\underline{\mathbb{Z}}_X^b \to \mathcal{K}_{0,X}$$

induced, on the level of presheaves, by sending 1 to 1. One can check, by passing to stalks, that this morphism is therefore an isomorphism. Hence one also has an isomorphism $\pi_*\underline{\mathbb{Z}}_X^b \cong \pi_*\mathcal{K}_{0,X}$.

Remark 4.1. The functor $\underline{\mathbb{Z}}_X^b$ is representable by a group X-scheme [Sta21, Tag 03P5] and, thus, $\underline{\mathbb{Z}}_X^b$ is also a sheaf in the étale topology. Moreover, for any X-scheme $Y \to X$, one can identify $\underline{\mathbb{Z}}_X^b(Y)$ with the group of locally constant functions from the topological space Y to $\mathbb{Z}$.

Theorem 4.2. *Let k be a field and let X/k be geometrically connected. Then there is a natural isomorphism*

$$\pi_*\mathcal{K}_{0,X} \cong \mathrm{Hom}_k(-, \mathbb{Z}),$$

where we write $\mathbb{Z}$ for the constant group k-scheme, locally of finite type over k, associated to the abstract group $\mathbb{Z}$.

Proof. Since X is geometrically connected we have that

$$\pi_*\mathcal{K}_{0,X}(k) = \mathrm{H}^0(X, \underline{\mathbb{Z}}_X) = \mathbb{Z}.$$

Let 1_k denote the positive generator for this group. For a k-scheme T, we write 1_T for the image of 1_k under the induced homomorphism

$$\pi_*\mathcal{K}_{0,X}(k) \to \pi_*\mathcal{K}_{0,X}(T)$$

associated to the structure map of T. Let $\mathbb{Z}_k^b$ be the constant presheaf on the big Zariski site of k associated to the group $\mathbb{Z}$. There is then an induced natural transformation

$$\mathbb{Z}_k^b \to \pi_*\mathcal{K}_{0,X}$$

sending the element $1 \in \mathbb{Z}_k^b(T)$ to $1_T \in \pi_*\mathcal{K}_{0,X}(T)$. Since the rightmost functor is a sheaf for the Zariski topology, there is an associated natural transformation

$$\underline{\mathbb{Z}}_k^b \to \pi_*\mathcal{K}_{0,X}$$

which identifies on a scheme T/k with the map

$$\underline{\mathbb{Z}}_k^b(T) = \mathrm{H}^0(T, \underline{\mathbb{Z}}_T) \to \mathrm{H}^0(X_T, \underline{\mathbb{Z}}_{X_T}) = \pi_*\mathcal{K}_{0,X}(T)$$

induced by projecting $X_T \to T$. Since X is geometrically connected, the projection $X_T \to T$ induces a bijection on connected components [Sta21, Tag 0385] and, consequently, this map is an isomorphism. ☐

Remark 4.3. Assume that $\pi : X \to k$ is a separated and finite type k-scheme. Then there exists an étale k-scheme $\pi_0(X/k)$ with a canonical faithfully flat map $X \to \pi_0(X/k)$ having the property that, for any field extension F/k, the F-points of $\pi_0(X/k)$ are in bijection with the geometrically connected components of X_F [DG70, Ch. I, §4, n° 6].

The sheafification of the constant presheaf $\mathbb{Z}^b_{\pi_0(X/k)}$ associated to $\mathbb{Z}$ on the big Zariski site for $\mathsf{Sch}/\pi_0(X/k)$ is representable [Sta21, Tag 03P5]. If $\mathbb{Z}_{\pi_0(X/k)}$ is the associated representing scheme for this functor, then the Weil restriction $\mathrm{Res}_{\pi_0(X/k)/k}(\mathbb{Z}_{\pi_0(X/k)})$, from $\pi_0(X/k)$ to k, exists as a k-scheme [BLR90, 7.6, Theorem 4]. Moreover, for any k-scheme T/k there are natural identifications

$$\mathrm{Hom}_k(T, \mathrm{Res}_{\pi_0(X/k)/k}(\mathbb{Z}_{\pi_0(X/k)})) = \mathrm{Hom}_{\pi_0(X/k)}(T_{\pi_0(X/k)}, \mathbb{Z}_{\pi_0(X/k)})$$

and the latter set is canonically the set of all Zariski locally constant maps from the space $\pi_0(X/k) \times_k T$ to the abstract group $\mathbb{Z}$.

For any k-scheme T/k, composition with the map $X \times_k T \to T_{\pi_0(X/k)}$ yields a map

$$\mathrm{Hom}_k(T, \mathrm{Res}_{\pi_0(X/k)/k}(\mathbb{Z}_{\pi_0(X/k)})) \to \mathrm{H}^0(X_T, \underline{\mathbb{Z}}_{X_T}) = \pi_*\mathcal{K}_{0,X}(T).$$

As these maps are natural in the argument T/k, they define a natural transformation between the functors $\mathrm{Hom}_k(-, \mathrm{Res}_{\pi_0(X/k)/k}(\mathbb{Z}_{\pi_0(X/k)}))$ and $\pi_*\mathcal{K}_{0,X}$. One can then check that this natural transformation is an isomorphism, cf. [BW19, Proposition 1.1].

We next consider the Zariski sheaf of units $\mathbb{G}_{m,X}$ on the site Sch/X. The sheaf $\mathbb{G}_{m,X}$ is known to be representable by a group scheme which we also denote by $\mathbb{G}_{m,X}$. We can canonically identify $\mathcal{K}_{1,X}$ with $\mathbb{G}_{m,X}$.

Theorem 4.4. *Let k be a field and let $\pi : X \to k$ be a proper scheme. Set $A = \mathrm{H}^0(X, \mathcal{O}_X)$ and write $S = \mathrm{Spec}(A)$. Then the functor $\pi_*\mathcal{K}_{1,X}$ is representable by the group k-scheme $\mathrm{Res}_{S/k}(\mathbb{G}_{m,S})$.*

Proof. We first remark that, since A is a finite k-algebra as X is proper [Sta21, Tag 02O6], the Weil restriction $\mathrm{Res}_{S/k}(\mathbb{G}_{m,S})$ exists as a k-scheme, see [BLR90, 7.6, Theorem 4]. Now there exists a canonically defined factorization of π into a composition

$$X \xrightarrow{\phi} S \xrightarrow{\rho} k$$

and we use this factorization to define a natural transformation

$$\mathrm{Res}_{S/k}(\mathbb{G}_{m,S}) \to \pi_*\mathbb{G}_{m,X}.$$

For any scheme T/k one has natural identifications

$$\pi_*\mathbb{G}_{m,X}(T) = \mathrm{Hom}_X(T_X, \mathbb{G}_{m,X}) = \mathcal{O}_{T_X}(T_X)^\times,$$

where $\mathcal{O}_{T_X}(T_X)^\times$ is the group of units of the ring $\mathcal{O}_{T_X}(T_X)$, and

$$\mathrm{Hom}_k(T, \mathrm{Res}_{S/k}(\mathbb{G}_{m,S})) = \mathrm{Hom}_S(T_S, \mathbb{G}_{m,S}) = \mathcal{O}_{T_S}(T_S)^\times.$$

We take as our definition of a natural transformation the map

$$\pi_*\mathbb{G}_{m,X}(T) \to \mathrm{Hom}_k(T, \mathrm{Res}_{S/k}(\mathbb{G}_{m,S}))$$

coming from taking global sections of the induced map $\mathcal{O}_{T_S} \to \phi_{T*}\mathcal{O}_{T_X}$. But note that since $X \to S$ is proper, and $T_S \to S$ is flat, the morphism $\mathcal{O}_{T_S} \to \phi_{T*}\mathcal{O}_{T_X}$ is an isomorphism [Sta21, Tag 02KH]. $\square$

Lastly, we observe that due to the identification of $\mathcal{K}_{1,X}$ and $\mathbb{G}_{m,X}$, there is a canonical isomorphism between $R^1\pi_*\mathcal{K}_{1,X}$ and $R^1\pi_*\mathbb{G}_{m,X}$. The latter of these sheaves is well-known to be isomorphic with the relative Zariski Picard functor $\mathrm{Pic}_{X/k,(Zar)}$, see [Kle05, Remark 9.2.11]. If X is proper, then the Picard functor $(R^1\pi_*\mathcal{K}_{1,X})_{fppf} \cong \mathrm{Pic}_{X/k,(fppf)}$ is representable by a scheme locally of finite type over k, see [Mur64, II.15, Theorem 2] and also [Oor62].

More generally, if S is any scheme and $\pi : X \to S$ is an S-scheme then $R^1\pi_*\mathcal{K}_{1,X}$ is still canonically identifiable with the Zariski Picard functor $\mathrm{Pic}_{X/S,(Zar)}$ by [Kle05, Remark 9.2.11]. Representability of the sheaf $(R^1\pi_*\mathcal{K}_{1,X})_{fppf}$ has been studied in many places, see for example [Art69, §7] and the end of [Kle05, §9.4] for a survey.

5. Effective pro-representability

Let S be a fixed but arbitrary scheme and let $\mathcal{F} : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ be any functor. For any field F, and for any F-point $s : \mathrm{Spec}(F) \to S$, we write $\mathcal{F}|_s : (\mathsf{Sch}/F)^{op} \to \mathsf{Set}$ for the restriction of $\mathcal{F}$ to Sch/F by s. Given an element $\xi_0 \in \mathcal{F}|_s(F)$, one can define deformation functors

$$\mathrm{Def}_{\xi_0}(\mathcal{F}|_s) : \mathsf{Art}_F \to \mathsf{Set} \tag{24}$$

from the category Art_F, whose objects are all local artinian F-algebras $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A \cong F$ and with morphisms between two objects being local F-algebra homomorphisms, by setting

$$\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)(A) = \{\xi_A \in \mathcal{F}|_s(\mathrm{Spec}(A)) : \mathcal{F}|_s(\iota)(\xi_A) = \xi_0\}, \tag{25}$$

where $\iota : \mathrm{Spec}(A/\mathfrak{m}) \to \mathrm{Spec}(A)$ is the closed immersion associated with the canonical quotient.

The functor $\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)$ is said to be *pro-representable* if there is a complete local noetherian F-algebra $(R, \mathfrak{m}_R)$ such that the quotients $R/\mathfrak{m}_R^t$ are objects of Art_F for all $t \geq 1$ together with a natural bijection

$$\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)(A) = \mathrm{Hom}_{\text{local } F\text{-alg}}(R, A) \tag{26}$$

for all objects $(A, \mathfrak{m}_A)$ of Art_F. For simplicity, we will also say that the functor $\mathcal{F}$ is pro-representable if for every field F, for every finite type F-point $s : \mathrm{Spec}(F) \to S$, and for every element $\xi_0 \in \mathcal{F}|_s(F)$ the functor $\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)$ is pro-representable.

Let $\widehat{\mathsf{Art}}_F$ be the category whose objects are all of the local noetherian F-algebras $(A, \mathfrak{m}_A)$ such that A is $\mathfrak{m}_A$-adically complete and $A/\mathfrak{m}_A^t$ is an object of Art_F for all $t \geq 1$; morphisms of this category are local homomorphisms of F-algebras. The category Art_F is a full subcategory of $\widehat{\mathsf{Art}}_F$ and there are two natural ways to extend the above functor of deformations of ξ_0 to this larger category. First, we can simply consider the restriction of $\mathcal{F}|_s$ to this larger collection of rings which we denote

$$\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s) : \widehat{\mathsf{Art}}_F \to \mathsf{Set};$$

these functors are defined identically with (25) but, for objects $(A, \mathfrak{m}_A)$ of the larger category $\widehat{\mathsf{Art}}_F$.

Second, we can consider the completed functors

$$\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s) : \widehat{\mathsf{Art}}_F \to \mathsf{Set}$$

defined on a local F-algebra $(A, \mathfrak{m}_A)$ of the category $\widehat{\mathsf{Art}}_F$ by

$$\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A) := \varprojlim_t \mathrm{Def}_{\xi_0}(\mathcal{F}|_s)(A/\mathfrak{m}_A^t),$$

with associated maps those that are canonically induced in the limit. Note that, since the Hom functor commutes with limits in the second argument, the functor $\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)$ is pro-representable if and only if the completed functor $\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)$ is genuinely representable, i.e. $\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)$ is pro-representable if and only if there is a complete local noetherian F-algebra $(R, \mathfrak{m}_R)$ of $\widehat{\mathsf{Art}}_F$ and a natural bijection

$$\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A) = \mathrm{Hom}_{\text{local } F\text{-alg}}(R, A) \tag{27}$$

for all objects $(A, \mathfrak{m}_A)$ of $\widehat{\mathsf{Art}}_F$.

Remark 5.1. Let $\pi : X \to S$ be an S-scheme and let τ be any topology that is at least as fine as the Zariski topology on Sch/X. If $\mathcal{F}$ is an abelian sheaf for the τ-topology then, because of Remark 2.8, a higher direct image functor $(R^i\pi_*\mathcal{F})_\tau$ is pro-representable if and only if for every field F and, for every finite type F-point $s : \mathrm{Spec}(F) \to S$, all of the functors $(R^i\pi_{s*}(\mathcal{F}|_{X_s}))_\tau$ are pro-representable. In other words, the pro-representability of a higher direct image functor is implied by the pro-representability of the higher direct image functor of the fibers.

For any complete local noetherian F-algebra $(A, \mathfrak{m}_A)$ of $\widehat{\mathsf{Art}}_F$, we say that elements of the set $\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A)$ are *formal deformations of* ξ_0 *over* A. A formal deformation is said to be *effective* if it is in the image of the canonical morphism

$$\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s)(A) \to \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A). \tag{28}$$

As a final point of terminology, we say that $\mathcal{F} : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ is *effectively pro-representable* if for every field F, for every finite type F-point $s : \mathrm{Spec}(F) \to S$, and for every element $\xi_0 \in \mathcal{F}|_s(F)$, the functor $\mathrm{Def}_{\xi_0}(\mathcal{F}|_s)$ is both pro-representable and if the formal deformation of ξ_0 associated to the identity, under the bijection from (27), is effective.

Lemma 5.2. *Let* $\mathcal{F} : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ *be given and suppose that* $\mathcal{F}$ *is effectively pro-representable. Then for every complete local noetherian* F*-algebra* A *of* $\widehat{\mathsf{Art}}_F$*, for every finite type* F*-point* $s : \mathrm{Spec}(F) \to S$*, and for every element* $\xi_0 \in \mathcal{F}|_s(F)$*, the morphism of (28) is surjective.*

Proof. Let R be the complete local Noetherian F-algebra representing the functor $\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)$. Let A be as above and let $\xi \in \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A)$ be given. There's then a morphism $f : R \to A$ so that the following diagram commutes and with $\widehat{\mathcal{F}|_s(f)}(\mathrm{id}_R) = \xi$.

$$\begin{array}{ccc}
\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s)(A) & \longrightarrow & \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(A) \\
{\scriptstyle \mathcal{F}|_s(f)}\uparrow & & \uparrow{\scriptstyle \widehat{\mathcal{F}|_s(f)}} \\
\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s)(R) & \longrightarrow & \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(R)
\end{array}$$

The claim now follows from the assumption that id_R is effective. ☐

Lemma 5.3. *Let* $\mathcal{F} : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ *be a pro-representable functor and suppose that for every field* F*, for every* F*-point* $s : \mathrm{Spec}(F) \to S$ *of* S *of finite type, and for every element* $\xi_0 \in \mathcal{F}|_s(F)$ *the map*

$$\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s)(R) \to \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(R) \to \mathrm{Def}_{\xi_0}(\mathcal{F}|_s)(R/\mathfrak{m}_R^2)$$

is surjective for $(R, \mathfrak{m}_R)$ *the complete local ring representing* $\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)$*. Then* $\mathcal{F}$ *is effectively pro-representable.*

Proof. The claim to be proved, in this notation, is that the universal formal deformation of ξ_0 (i.e. the formal deformation associated to the identity) is effective. However, under the assumptions of the lemma, Artin has shown in [Art69, Section 1] the more general statement that an arbitrary versal formal deformation of ξ_0 is effective. ☐

5.1. **Abelian functors.** From now on, we fix an arbitrary field k (to be used as a base) and we write $\mathcal{F} : (\mathsf{Sch}/k)^{op} \to \mathsf{Ab}$ for an arbitrary functor valued in the category abelian groups. The following lemma says that checking pro-representability of $\mathcal{F}$ can reduced to checking at the identity element; the proof of the lemma is immediate from the definitions, but we include it here for completeness.

Lemma 5.4. *Let* F/k *be a finite field extension. Let* $0_F \in \mathcal{F}(F)$ *be the identity of this group and let* $\xi \in \mathcal{F}(F)$ *be an arbitrary element. Then, if* $\mathrm{Def}_{0_F}(\mathcal{F})$ *is pro-representable, the functor* $\mathrm{Def}_\xi(\mathcal{F})$ *is pro-representable as well.*

Proof. For any complete local noetherian F-algebra $(R, \mathfrak{m}_R)$ of $\widehat{\mathsf{Art}}_F$, there are natural homomorphisms $F \to R \to F$, namely the F-algebra map and the quotient map, which compose to the identity map on F. At the functor level, the induced maps

$$\mathcal{F}(F) \to \mathcal{F}(\mathrm{Spec}(R)) \to \mathcal{F}(F)$$

realizes a canonical inclusion $\mathcal{F}(F) \subset \mathcal{F}(\mathrm{Spec}(R))$ which splits the map induced by the quotient. Since morphisms in $\widehat{\mathsf{Art}}_F$ are local, this inclusion is independent of the choice of the ring R. More precisely, given any ring map $\rho : R \to R'$ between R and another object $(R', \mathfrak{m}_{R'})$ of $\widehat{\mathsf{Art}}_F$, the following diagram commutes

$$\begin{array}{ccccc} \mathcal{F}(F) & \longrightarrow & \mathcal{F}(\mathrm{Spec}(R)) & \longrightarrow & \mathcal{F}(F) \\ \| & & \downarrow{\scriptstyle \mathcal{F}(\rho)} & & \| \\ \mathcal{F}(F) & \longrightarrow & \mathcal{F}(\mathrm{Spec}(R')) & \longrightarrow & \mathcal{F}(F). \end{array}$$

Suppose then that there is a complete local noetherian F-algebra R and a canonical isomorphism of functors

$$\mathrm{Def}_{0_F}(\mathcal{F})(-) = \mathrm{Hom}_{\text{local } F\text{-alg}}(R, -).$$

Subtracting $\xi \in \mathcal{F}(F)$ then yields a canonical isomorphism

$$\mathrm{Def}_{\xi}(\mathcal{F})(-) \xrightarrow{\xi' \mapsto \xi' - \xi} \mathrm{Def}_{0_F}(\mathcal{F})(-) = \mathrm{Hom}_{\text{local } F\text{-alg}}(R, -)$$

as can be readily checked by evaluating the maps at any local artinian F-algebra $(A, \mathfrak{m}_A)$ of Art_F. □

Checking the effectivity of pro-representability can also be reduced to checking the appropriate claim at the identity element of the group.

Lemma 5.5. *Let F/k be a finite field extension. Let $0_F \in \mathcal{F}(F)$ be the identity of this group and let $\xi \in \mathcal{F}(F)$ be an arbitrary element. Assume that $\mathcal{F}$ is pro-representable.*

If the formal deformation associated to the identity map in $\widehat{\mathrm{Def}}_{0_F}(\mathcal{F})$ is effective, then the formal deformation associated to the identity map in $\widehat{\mathrm{Def}}_{\xi}(\mathcal{F})$ is effective.

Proof. Making identifications as in Lemma 5.4, the claim follows from the commutativity of the square below

$$\begin{array}{ccc} \mathrm{Def}^c_{0_F}(\mathcal{F})(A) & \longrightarrow & \widehat{\mathrm{Def}}_{0_F}(\mathcal{F})(A) \\ \downarrow{\scriptstyle \xi_A \mapsto \xi_A + \xi} & & \downarrow{\scriptstyle \xi_A \mapsto \xi_A + \xi} \\ \mathrm{Def}^c_{\xi}(\mathcal{F})(A) & \longrightarrow & \widehat{\mathrm{Def}}_{\xi}(\mathcal{F})(A) \end{array}$$

for any complete local noetherian ring A of $\widehat{\mathsf{Art}}_F$. Here the horizontal arrows are the map from (28). □

5.2. **Higher direct images of K-theory sheaves.** Throughout this subsection we study the pro-representability and the effective pro-representability of the higher direct images $R^i\pi_*\mathcal{K}_{n,X}$ associated to an S-scheme $\pi : X \to S$. Here we present our main result, Theorem 5.15, which proves the effectivity of pro-representability in some new cases. As a corollary to this theorem, and of the results above, we obtain an algebraizability result for the higher direct image functors.

Proposition 5.6. *Let $k/\mathbb{Q}$ be an algebraic extension of $\mathbb{Q}$, let S/k be a finite type k-scheme, and let $\pi : X \to S$ be a quasi-compact and quasi-separated S-scheme. Then the following statements hold.*

(1) Suppose that, for every field F and for every finite-type F-point $s : \mathrm{Spec}(F) \to S$, the fiber $\pi^{-1}(s) = X_s$ is a smooth, proper, and geometrically connected variety satisfying

$$\mathrm{H}^i(X_s, \mathcal{O}_{X_s}) = \mathrm{H}^{i+1}(X_s, \mathcal{O}_{X_s}) = 0.$$

Then both $R^i\pi_*\mathcal{K}^M_{2,X}$ and $(R^i\pi_*\mathcal{K}^M_{2,X})_{ét}$ are pro-representable. Further, these functors are pro-represented at s by the F-algebra $R = F[[t_1, ..., t_r]]$ where $r = \dim_F \mathrm{H}^i(X_s, \Omega^1_{X_s})$.

(2) Suppose that, for every field F and for every finite-type F-point $s : \mathrm{Spec}(F) \to S$, the fiber $\pi^{-1}(s) = X_s$ is a smooth, proper, and geometrically connected variety satisfying both

$$\mathrm{H}^i(X_s, \mathcal{O}_{X_s}) = \mathrm{H}^{i+1}(X_s, \mathcal{O}_{X_s}) = \mathrm{H}^{i+2}(X_s, \mathcal{O}_{X_s}) = 0$$

and also

$$\mathrm{H}^i(X_s, \Omega^1_{X_s}) = \mathrm{H}^{i+1}(X_s, \Omega^1_{X_s}) = 0.$$

Then both $R^i\pi_*\mathcal{K}^M_{3,X}$ and $(R^i\pi_*\mathcal{K}^M_{3,X})_{ét}$ are pro-representable. Further, these functors are pro-represented at s by the F-algebra $R = F[[t_1, ..., t_r]]$ where $r = \dim_F \mathrm{H}^i(X_s, \Omega^2_{X_s})$.

Proof. It suffices, by Remark 5.1, to prove the same claim for the each of the fibers. In this way, we reduce to the case where the base scheme S is a field which, by abuse of notation, we call k. We write $\pi : X \to k$ for the structure map and let F/k be any finite field extension of k. We first prove the claim of (1) only for the functor $R^i\pi_*\mathcal{K}^M_{2,X}$. Then, separately, we give an analogous proof for the étale sheafification $(R^i\pi_*\mathcal{K}^M_{2,X})_{ét}$. The proof of the claim in (2) will be similar.

By Lemma 5.4, it suffices to check that the functor of deformations $\mathrm{Def}_{0_F}(R^i\pi_*\mathcal{K}^M_{2,X})$ of the identity $0_F \in R^i\pi_*\mathcal{K}^M_{2,X}(F)$ is pro-representable. Now it's an immediate consequence of the construction of the higher direct image functors, see Remark 2.5, that the deformation functor $\mathrm{Def}_{0_F}(R^i\pi_*\mathcal{K}^M_{2,X}) : \mathsf{Art}_F \to \mathsf{Set}$ is isomorphic to the tangent functor $\mathscr{T}^{i,2}_X : \mathsf{Art}_F \to \mathsf{Set}$ defined by

$$\mathscr{T}^{i,2}_X(A) = \ker\left(\mathrm{H}^i(X_T, \mathcal{K}^M_{2,X_T}) \to \mathrm{H}^i(X_F, \mathcal{K}^M_{2,X_F})\right)$$

where $T = \mathrm{Spec}(A)$ is a local artinain F-algebra with residue field F. With the given assumptions on X, the latter functor was shown to be pro-representable by Bloch in [Blo75] where, moreover, it was shown that there is a natural isomorphism $\mathscr{T}^{i,2}_X(A) = \mathrm{H}^i(X_F, \Omega^1_{X_F}) \otimes_F \mathfrak{m}_A$ for the maximal ideal $\mathfrak{m}_A$ of A.

Similarly, as a result of the description of Remark 3.3, the functor of deformations $\mathrm{Def}_{0_F}((R^i\pi_*\mathcal{K}^M_{2,X})_{ét})$ of the identity $0_F \in (R^i\pi_*\mathcal{K}^M_{2,X})_{ét}(F)$

is canonically isomorphic to the functor $\mathscr{T}^{i,2}_{X,ét} : \mathsf{Art}_F \to \mathsf{Set}$ defined by

$$\mathscr{T}^{i,2}_{X,ét}(A) = \ker\left(\mathrm{H}^i(X_T, \mathcal{K}^M_{2,X_T})^{G_F} \to \mathrm{H}^i(X_{F^s}, \mathcal{K}^M_{2,X_{F^s}})^{G_F}\right)$$

where F^s is a fixed separable closure of F, where $G_F = \mathrm{Gal}(F^s/F)$ is the absolute Galois group, and where $T = \mathrm{Spec}(A \otimes_F F^s)$. Now the result again follows from Bloch's result in [Blo75] and, further, there is a canonical identification $\mathscr{T}^{i,2}_X(A) = \mathscr{T}^{i,2}_{X,ét}(A)$.

The same idea works to show the claim of (2), using the main result of [Mac23] and the functors $\mathscr{T}^{i,3}_X$ defined there. □

Example 5.7. Let k be an algebraic extension of $\mathbb{Q}$ and suppose that $\pi : X \to k$ is a smooth, proper, and irreducible surface with geometric genus $p_g(X) = \dim \mathrm{H}^0(X, \Omega^2_X) = 0$. Then the assumptions of (1) hold for $i = 2$, hence both $R^2\pi_*\mathcal{K}^M_{2,X}$ and $(R^2\pi_*\mathcal{K}^M_{2,X})_{ét}$ are pro-representable. If, moreover, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ then this pro-representability is trivially effective since in this case $\mathrm{H}^2(X, \Omega^1_X) = 0$.

Example 5.8. Let k be an algebraic extension of $\mathbb{Q}$ and suppose that $\pi : X \to k$ is a smooth, proper, and irreducible threefold with both $\mathrm{H}^3(X, \mathcal{O}_X) = 0$ and $\mathrm{H}^2(X, \mathcal{O}_X) = 0$. Then the assumptions of (2) hold for $i = 3$, hence both $R^3\pi_*\mathcal{K}^M_{3,X}$ and $(R^3\pi_*\mathcal{K}^M_{3,X})_{ét}$ are pro-representable. If, moreover, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ then this pro-representability is trivially effective as well since in this case $\mathrm{H}^3(X, \Omega^2_X) = 0$.

Example 5.9. If k is an algebraic extension of $\mathbb{Q}$ and if $\pi : X \to k$ is a smooth, proper, and irreducible scheme such that the Chow motive of X is a direct sum of Tate motives, then the Hodge numbers $h^{p,q} = 0$ vanish for all (p, q) with $p \neq q$, cf. [Tot16, Theorem 4.1]. More generally, to get vanishing of the Hodge numbers it is sufficient to assume that the motive of X_F over an algebraic closure $F = \bar{\mathbb{Q}}$ is mixed Tate [Tot16, Corollary 7.3]. This gives examples where both (1) and (2) hold.

Example 5.10. Over an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, there are smooth and proper threefolds that are both irrational and unirational, see [AM72]. For such varieties the assumptions of (1) hold for any $i \geq 1$ by [ACTP17, Proposition 1.11 and Proposition 1.8]. Moreover, because of [ACTP17, Theorem 1.4], for any such threefold X there exists a field $F/\bar{\mathbb{Q}}$ with $\ker(\deg : \mathrm{CH}_0(X_F) \to \mathbb{Z}) \neq 0$.

Remark 5.11. The following observation will be used often below. Suppose that k is a field and $R = k[[t_1, ..., t_r]]$ is a ring of formal power series over k in finitely many variables. Let X be a smooth scheme, geometrically connected and quasi-compact over k. Then the product $X' = X \times_k \mathrm{Spec}(R)$ is a connected, regular, and excellent Noetherian scheme of finite Krull dimension. Indeed, if $U = \mathrm{Spec}(B)$ is an affine open subset of X, then B is a finitely generated k-algebra and $B \otimes_k R$ is Noetherian by Hilbert's basis theorem. Thus X' is locally Noetherian and the composition $X' \to \mathrm{Spec}(R) \to k$ is quasi-compact, so X' is locally Noetherian and quasi-compact, hence Noetherian.

Since X is geometrically connected and R is integral, it follows that X' is connected, [Sta21, Tag 0385]. One can compute the dimension of X' in terms of the dimension of X and R, [Sta21, Tag 0AFF]; note that this also gives a bound for the dimension of the local rings of X', [Sta21, Tag 04MU]. Excellence of X' follows from [Sta21, Tag 07QW].

To see that X' is regular, let $x \in X'$ be a point. Let $y \in \mathrm{Spec}(R)$ be the image of the point x under the projection $X' \to \mathrm{Spec}(R)$, and write $\mathfrak{m}_y \subset \mathcal{O}_{\mathrm{Spec}(R),y}$ for the maximal ideal of the local ring at y. The induced ring map $\mathcal{O}_{\mathrm{Spec}(R),y} \to \mathcal{O}_{X',x}$ is both local and, since X is smooth, flat. Since there exists a canonical isomorphism

$$\mathcal{O}_{X',x}/\mathfrak{m}_y\mathcal{O}_{X',x} \cong \mathcal{O}_{X'_y,x},$$

between the fiber over y of the local ring of x in X' and the local ring of x in the fiber over y, and since X is smooth, the local ring $\mathcal{O}_{X'_y,x} \cong \mathcal{O}_{X',x}/\mathfrak{m}_y\mathcal{O}_{X',x}$ is regular. As $\mathcal{O}_{\mathrm{Spec}(R),y}$ is also regular, it follows from [Mat86, Theorem 23.7] that $\mathcal{O}_{X',x}$ is regular too.

In particular, the Gersten conjecture for Quillen's K-theory [Pan03, Theorem A], and the Gersten conjecture for Milnor K-theory [Ker09, Theorem 7.1] when k has enough elements, are both known to hold for the local rings of X'.

Lemma 5.12. *Fix an algebraic extension $k/\mathbb{Q}$ and let $\pi : X \to k$ be a smooth, proper, and geometrically connected scheme. Let $\mathcal{E}$ be a finite rank locally free sheaf on X and $\varphi : \mathbb{P}(\mathcal{E}) \to k$ be the structure map of the associated projective bundle. Then $R^2\pi_*\mathcal{K}^M_{2,X}$ is pro-representable (resp. effectively pro-representable) if and only if $R^2\varphi_*\mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})}$ is pro-representable (resp. effectively pro-representable).*

The above statement also holds replacing all Zariski sheaves with their étale sheafifications.

Proof. It suffices to work only over the base field k, noting that $k/\mathbb{Q}$ is arbitrary. Note that $R^2\pi_*\mathcal{K}^M_{2,X}$ is pro-representable if and only if

$$\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^3(X, \mathcal{O}_X) = 0 \tag{29}$$

by [Blo75, Theorem (0.2)] and Lemma 5.4. Now (29) holds if and only if there is the vanishing of cohomology

$$\mathrm{H}^2(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}) = \mathrm{H}^3(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}) = 0$$

which holds if and only if $R^2\varphi_*\mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})}$ is pro-representable by [Blo75, Theorem (0.2)] and Lemma 5.4 again. We note that the same is true for the étale sheafifications of these functors.

Let $R = k[[t_1, ..., t_r]]$ be a power series ring in finitely many variables with maximal ideal $\mathfrak{m}_R$. Set $S = \mathrm{Spec}(R)$ and $S_t = \mathrm{Spec}(R/\mathfrak{m}_R^t)$. Then, by Remark 5.11, the Gersten conjecture holds for the local rings of both X_S and $\mathbb{P}(\mathcal{E})_S$ so that

$$\mathrm{H}^2(X_S, \mathcal{K}^M_{2,X_S}) \cong \mathrm{CH}^2(X_S) \quad \text{and} \quad \mathrm{H}^2(\mathbb{P}(\mathcal{E})_S, \mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})_S}) \cong \mathrm{CH}^2(\mathbb{P}(\mathcal{E})_S)$$

where for any equidimensional k-scheme Y we write $\mathrm{CH}^2(Y)$ for the group of codimension-2 cycles on Y modulo rational equivalence.

Now let $0_k \in R^2\pi_*\mathcal{K}^M_{2,X}(k)$ be the group identity and suppose that $\mathrm{Def}_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})$ is pro-representable. Then by [Blo75, Theorem (0.2)], the functor $\mathrm{Def}_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})$ is isomorphic to the functor assigning to an artinian local k-algebra $(A, \mathfrak{m}_A)$ the k-vector space $\mathrm{H}^2(X, \Omega^1_X) \otimes_k \mathfrak{m}_A$. Thus there is a commutative diagram as below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Def}^c_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})(R) & \longrightarrow & \mathrm{CH}^2(X_S) & \longrightarrow & \mathrm{CH}^2(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{H}^2(X, \Omega^1_X) \otimes_k \mathfrak{m}_R & \longrightarrow & \varprojlim_t \mathrm{H}^2(X_{S_t}, \mathcal{K}^M_{2,X_{S_t}}) & \longrightarrow & \mathrm{H}^2(X, \mathcal{K}^M_{2,X}) & \longrightarrow & 0
\end{array}$$

Here the surjectivity of the bottom row follows from [Sta21, Tag 0598]. Similarly, there is a commutative diagram for $\mathbb{P}(\mathcal{E})$ as so:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Def}^c_{0_k}(R^2\varphi_*\mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})})(R) & \longrightarrow & \mathrm{CH}^2(\mathbb{P}(\mathcal{E})_S) & \longrightarrow & \mathrm{CH}^2(\mathbb{P}(\mathcal{E})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{H}^2(\mathbb{P}(\mathcal{E}), \Omega^1_{\mathbb{P}(\mathcal{E})}) \otimes_k \mathfrak{m}_R & \longrightarrow & \varprojlim_t \mathrm{H}^2(\mathbb{P}(\mathcal{E})_{S_t}, \mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})_{S_t}}) & \longrightarrow & \mathrm{H}^2(\mathbb{P}(\mathcal{E}), \mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})}) & \longrightarrow & 0.
\end{array}$$

The bottom rows in the diagrams above are canonically right-split by the pull-back along the structure maps for X and $\mathbb{P}(\mathcal{E})$ respectively. Using the projection map $\mathbb{P}(\mathcal{E}) \to X$, the two diagrams above can be compared (via pull-back) and this comparison respects these splittings. If the rank of $\mathcal{E}$ is 1, then the comparison is an isomorphism everywhere. Otherwise, if the rank of $\mathcal{E}$ is greater than 1, then there is a diagram of corresponding cokernels:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Def}^c_{0_k}(\mathrm{Pic}_{X/k,(Zar)})(R) & \longrightarrow & \mathrm{Pic}(X_S) \oplus A & \longrightarrow & \mathrm{Pic}(X) \oplus A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{H}^1(X, \mathcal{O}_X) \otimes_k \mathfrak{m}_R & \longrightarrow & \varprojlim_t \mathrm{H}^1(X_{S_t}, \mathcal{K}^M_{1,X_{S_t}}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{K}^M_{1,X}) & \longrightarrow & 0.
\end{array}$$

Here the nonzero column on the right (and the middle object in the top row) can be identified with use of the projective bundle formula (and we have $A = 0$ if $\mathrm{rank}(\mathcal{E}) = 2$ and $A = \mathbb{Z}$ if $\mathrm{rank}(\mathcal{E}) \geq 3$); the identification of the top-left object follows from this. The identification of the bottom-left object seems to be well-known, and the middle term in the bottom row can be identified using these facts together with the splittings of the bottom rows of the two previous diagrams.

Altogether, this gives a commutative ladder with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Def}^c_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})(R) & \to & \mathrm{Def}^c_{0_k}(R^2\varphi_*\mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})})(R) & \to & \mathrm{Def}^c_{0_k}(\mathrm{Pic}_{X/k,(Zar)})(R) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(X, \Omega^1_X) \otimes_k \mathfrak{m}_R & \longrightarrow & \mathrm{H}^2(\mathbb{P}(\mathcal{E}), \Omega^1_{\mathbb{P}(\mathcal{E})}) \otimes_k \mathfrak{m}_R & \longrightarrow & \mathrm{H}^1(X, \mathcal{O}_X) \otimes_k \mathfrak{m}_R & \longrightarrow & 0. \end{array}$$

Here the rightmost vertical arrow is an isomorphism by Grothendieck's existence theorem, cf. [Sta21, Tag 089N]. Therefore, if either the left or the middle vertical arrow is a surjection, then the other is as well. By varying the power series ring R, it follows from Lemma 5.2 and Lemma 5.5 that $R^2\pi_*\mathcal{K}^M_{2,X}$ is effectively pro-representable if and only if $R^2\varphi_*\mathcal{K}^M_{2,\mathbb{P}(\mathcal{E})}$ is effectively pro-representable. The analogous theorem for the étale higher direct image functors is proved similarly by noting that, in each of the above diagrams, all splittings descend to Galois invariants. □

Remark 5.13. Lemma 5.12 has the following natural generalization. Assume that $R^k\pi_*\mathcal{K}^M_{k,X}$ is pro-representable for all integers k with $0 \leq k \leq n$ (resp. that these functors are effectively pro-representable). Then the functor $R^n\varphi_*\mathcal{K}^M_{n,\mathbb{P}(\mathcal{E})}$ is pro-representable (resp. effectively pro-representable). Conversely, if for an integer $n \geq 0$ each of the functors $R^n\varphi_*\mathcal{K}^M_{n,\mathbb{P}(\mathcal{E})}$ and $R^k\pi_*\mathcal{K}^M_{k,X}$ are pro-representable (resp. effectively pro-representable) for all integers k with $0 \leq k < n$, then $R^n\pi_*\mathcal{K}^M_{n,X}$ is pro-representable (resp. effectively pro-representable). There is also a natural étale version of this generalization.

For $n = 0$ and $n = 1$, the above statement is true and the proof follows lines similar to the proof of Lemma 5.12. When $n = 2$, this statement is the content of Lemma 5.12. For $n \geq 3$, the proof of this statement seems out of reach at the moment.

Lemma 5.14. *Fix an algebraic extension $k/\mathbb{Q}$ and let $\pi : X \to k$ be a smooth, proper, and geometrically connected scheme. Let $Z \subset X$ be a smooth subscheme of X and let $\varphi : \mathrm{Bl}_Z(X) \to k$ be the structure map of the blow-up of X along Z. Then $R^2\pi_*\mathcal{K}^M_{2,X}$ is pro-representable* *(resp. effectively pro-representable) if and only if* $R^2\varphi_*\mathcal{K}^M_{2,\mathrm{Bl}_Z(X)}$ *is pro-representable (resp. effectively pro-representable).*

The above statement also holds replacing all Zariski sheaves with their étale sheafifications.

Proof. As in the proof of Lemma 5.12, $R^2\pi_*\mathcal{K}^M_{2,X}$ is pro-representable if and only if

$$\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^3(X, \mathcal{O}_X) = 0 \tag{30}$$

by [Blo75, Theorem (0.2)] and Lemma 5.4. Now (30) holds if and only if there is the vanishing of cohomology

$$\mathrm{H}^2(\mathrm{Bl}_Z(X), \mathcal{O}_{\mathrm{Bl}_Z(X)}) = \mathrm{H}^3(\mathrm{Bl}_Z(X), \mathcal{O}_{\mathrm{Bl}_Z(X)}) = 0$$

(see e.g. [CR11] or [RYYY22]) which holds if and only if $R^2\varphi_*\mathcal{K}^M_{2,\mathrm{Bl}_Z(X)}$ is pro-representable by [Blo75, Theorem (0.2)] and Lemma 5.4 again. We note that the same is true for the étale sheafifications.

For the effectivity statement of the lemma, it suffices to assume $\mathrm{codim}_X(Z) \geq 2$. As before, let $R = k[[t_1, ..., t_r]]$ be a power series ring in finitely many variables with maximal ideal $\mathfrak{m}_R$. Set $S = \mathrm{Spec}(R)$. We write $\mathcal{N}_{Z/X}$ for the normal sheaf on Z of the inclusion $Z \subset X$ and we write $\Theta : \mathbb{P}(\mathcal{N}_{Z/X}) \to X$ for the associated projective bundle map. There's an isomorphism $\mathrm{Pic}(\mathbb{P}(\mathcal{N}_{Z/X})) \cong \Theta^*\mathrm{Pic}(Z) \oplus \mathbb{Z}c_1(\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Z/X})}(1))$. By the blow-up formula for regular embeddings [Ful98, Proposition 6.7 (e)] there is an exact sequence

$$0 \to \mathrm{CH}^2(X) \to \mathrm{CH}^2(\mathrm{Bl}_Z(X)) \to \mathrm{Pic}(\mathbb{P}(\mathcal{N}_{Z/X}))/A \to 0$$

where $A = 0$ if $\mathrm{codim}_X(Z) > 2$ or, if $\mathrm{codim}_X(Z) = 2$, then A is the infinite cyclic subgroup generated by $\Theta^*c_1(\mathcal{N}_{Z/X}) + c_1(\mathcal{O}_{\mathbb{P}(\mathcal{N}_{Z/X})}(1))$. Since Remark 5.11 applies to the triple $(X_S, Z_S, \mathrm{Bl}_{Z_S}(X_S))$ and, due to [Sta21, Tag 0E9J], the blow-up formula also provides an exact sequence

$$0 \to \mathrm{CH}^2(X_S) \to \mathrm{CH}^2(\mathrm{Bl}_{Z_S}(X_S)) \to \mathrm{Pic}(\mathbb{P}(\mathcal{N}_{Z_S/X_S}))/A_S \to 0$$

with A_S characterized by the codimension of $Z \subset X$ similarly.

If $\mathrm{codim}_X(Z) = 2$, then pulling back induces isomorphisms

$$\mathrm{Pic}(Z) \cong \mathrm{Pic}(\mathbb{P}(\mathcal{N}_{Z/X}))/A \quad \text{and} \quad \mathrm{Pic}(Z_S) \cong \mathrm{Pic}(\mathbb{P}(\mathcal{N}_{Z_S/X_S}))/A_S.$$

Now, regardless of the codimension of Z in X, the above sequences on Chow groups produce an exact sequence

$$0 \longrightarrow \mathrm{Def}^c_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})(R) \longrightarrow \mathrm{Def}^c_{0_k}(R^2\varphi_*\mathcal{K}^M_{2,\mathrm{Bl}_Z(X)})(R) \longrightarrow \mathrm{Def}^c_{0_k}(\mathrm{Pic}_{Z/k,(Zar)})(R) \longrightarrow 0.$$

If either of $R^2\pi_*\mathcal{K}^M_{2,X}$ or $R^2\varphi_*\mathcal{K}^M_{2,\mathrm{Bl}_Z(X)}$ are pro-representable, then pulling back along the map $\mathrm{Bl}_Z(X) \to X$ induces an exact commutative ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Def}^c_{0_k}(R^2\pi_*\mathcal{K}^M_{2,X})(R) & \longrightarrow & \mathrm{Def}^c_{0_k}(R^2\varphi_*\mathcal{K}^M_{2,\mathrm{Bl}_Z(X)})(R) & \longrightarrow & \mathrm{Def}^c_{0_k}(\mathrm{Pic}_{Z/k,(Zar)})(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(X,\Omega^1_X)\otimes_k \mathfrak{m}_R & \longrightarrow & \mathrm{H}^2(\mathrm{Bl}_Z(X),\Omega^1_{\mathrm{Bl}_Z(X)})\otimes_k \mathfrak{m}_R & \longrightarrow & \mathrm{H}^1(Z,\mathcal{O}_Z)\otimes_k \mathfrak{m}_R & \longrightarrow & 0. \end{array}$$

The rightmost vertical arrow is an isomorphism. Hence, if either of the left or the middle vertical arrows were surjective, then the other would be as well. This allows us to conclude as before. □

Theorem 5.15. *Fix an algebraic extension $k/\mathbb{Q}$ and let $\pi_X : X \to k$ and $\pi_Y : Y \to k$ be two smooth, proper, and geometrically connected k-schemes. Suppose that X and Y are stably birational over k.*

Then $(R^2\pi_{X*}\mathcal{K}^M_{2,X})_\tau$ is pro-representable (respectively effectively pro-representable) if and only if $(R^2\pi_{Y*}\mathcal{K}^M_{2,Y})_\tau$ is pro-representable (resp. effectively pro-representable) for either $\tau = \mathrm{Zar}, \acute{e}t$.

Proof. This follows immediately from Lemma 5.12, Lemma 5.14, and the Weak Factorization theorem over k, [Włod09, Theorem 0.0.1 (1)]. Namely, suppose $(R^2\pi_{X*}\mathcal{K}^M_{2,X})_\tau$ is either pro-representable or effectively pro-representable and let $\varphi_X : X \times \mathbb{P}^r \to k$ and $\varphi_Y : Y \times \mathbb{P}^s \to k$ be birationally equivalent k-schemes for some $r, s \geq 0$.

Then the higher push forward functor $(R^2\varphi_{X*}\mathcal{K}^M_{2,X\times\mathbb{P}^r})_\tau$ is either pro-representable or effectively pro-representable because of Lemma 5.12. Any birational equivalence between the two schemes $X\times\mathbb{P}^r$ and $Y\times\mathbb{P}^s$ can be factored into a sequence of blow-ups and blow-downs, by the Weak Factorization theorem, so that this implies $(R^2\varphi_{Y*}\mathcal{K}^M_{2,Y\times\mathbb{P}^s})_\tau$ is then pro-representable or effectively pro-representable by Lemma 5.14. We can then conclude that $(R^2\pi_{Y*}\mathcal{K}^M_{2,Y})_\tau$ is also pro-representable or effectively pro-representable by use of Lemma 5.12 again. □

Let $\mathcal{F} : (\mathsf{Sch}/k)^{op} \to \mathsf{Set}$ be an effectively pro-representable functor. Let F/k be a finite field extension and let $s : \mathrm{Spec}(F) \to \mathrm{Spec}(k)$ be the corresponding F-point. Then for any $\xi_0 \in \mathcal{F}|_s(F)$, there is a complete local F-algebra $(R, \mathfrak{m}_R)$ so that the functor $\widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)$ is representable by the F-algebra R. Moreover, the universal formal deformation of ξ_0 corresponding to the identity of R is in the image of the canonical map

$$\mathrm{Def}^c_{\xi_0}(\mathcal{F}|_s)(R) \to \widehat{\mathrm{Def}}_{\xi_0}(\mathcal{F}|_s)(R)$$

described in (28).

For any complete local F-algebra $(A, \mathfrak{m}_A)$ of the category $\widehat{\mathsf{Art}}_F$, we say that a formal deformation ξ_A of ξ_0 over A is *algebraizable* if there exists a finite type F-scheme X, a closed point $x \in X$ with $k(x) \cong F$, an element $\xi \in \mathcal{F}|_s(X)$, and an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong A$ between the completion of $\mathcal{O}_{X,x}$ with respect to the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and A such that ξ induces ξ_A, i.e. the image of ξ under the map

$$\mathcal{F}|_s(X) \to \mathcal{F}|_s(\mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_x^t))$$

coincides with the image of ξ_A for all $t \geq 1$.

We say that the functor $\mathcal{F}$ is *algebraizable* if for any triple (F, s, ξ_0) as above, the universal formal deformation of ξ_0 corresponding to the identity is algebraizable. As a consequence of Artin's Algebraization Theorem [Art69, Theorem 1.6], a sufficient condition for an effectively pro-representable functor $\mathcal{F}$ to be algebraizable is that $\mathcal{F}$ is locally of finite presentation. Thus, as an immediate corollary to Theorem 5.15, Proposition 3.1, and Artin's theorem we get:

Corollary 5.16. *Fix an algebraic extension $k/\mathbb{Q}$ and let $\pi_X : X \to k$ and $\pi_Y : Y \to k$ be two smooth, proper, and geometrically connected k-schemes. Suppose that X and Y are stably birational over k.*

Then, for either $\tau = \mathrm{Zar}$ or ét, $(R^2\pi_{X*}\mathcal{K}^M_{2,X})_\tau$ is algebraizable if and only if $(R^2\pi_{Y*}\mathcal{K}^M_{2,Y})_\tau$ is algebraizable. □

Example 5.17. In [BW19], Benoist and Wittenberg construct a group functor $\mathrm{CH}^2_{X/k,fppf}$, defined on the category of quasi-compact and quasi-separated k-schemes, for any smooth, proper, geometrically connected threefold X over a field k such that the map $\deg : \mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism for every field extension F/k. The functor $\mathrm{CH}^2_{X/k,fppf}$ is thought of as an analogue of the Picard functor for codimension 2-cycles and admits an isomorphism

$$\mathrm{CH}^2_{X/k,fppf}(\overline{k}) \cong \mathrm{CH}^2(X_{\overline{k}})$$

for any algebraic closure $\overline{k}/k$. The authors are able to show that for geometrically rational X, the group functor $\mathrm{CH}^2_{X/k,fppf}$ is representable by a smooth group scheme $\mathbf{CH}^2_{X/k}$ over k.

While Theorem 5.15 and Corollary 5.16 don't show representability, they are close in the sense that they can be used to verify conditions [0], [1] and [2] of Artin's representability criterion [Art69, Theorem 4.1] for the functors $(R^2\pi_*\mathcal{K}^M_{2,X})_{ét}$ in a handful of new cases. (To fully prove that $(R^2\pi_*\mathcal{K}^M_{2,X})_{ét}$ is representable by a group k-scheme, there is still condition [3], on relative representability, that needs to be checked.)

Compared to Benoist and Wittenberg's construction, our results can also be applied to schemes which do not have a universally trivial Chow group of dimension 0-cycles. For example, if S is an Enriques surface defined over $\mathbb{Q}$, then it follows from Theorem 5.15 that $(R^2\pi_*\mathcal{K}^M_{2,X})_{\acute{e}t}$ is algebraizable for the associated threefold product $\pi : X = S\times\mathbb{P}^1 \to \mathbb{Q}$. However, by Theorem [ACTP17, Theorem 1.4] and the remarks under [ACTP17, Proposition 1.8], the Chow group of dimension 0-cycles on X is not universally trivial.

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