THE GENERA OF SMOOTH CURVES IN A PROJECTIVE VARIETY

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ABSTRACT. In this note we study the set $\Sigma_g(X)$ of nonnegative integers that appear as the genera of smooth curves on a smooth and projective variety X. We establish some basic results for these sets including: over an arbitrary field, $\Sigma_g(X)$ is infinite if and only if $\dim(X) \geq 2$; over an infinite field, $\Sigma_g(X)$ is dense in the set of nonnegative integers whenever X contains a smooth surface S admitting a nonconstant morphism to $\mathbb{P}^1$.

Notation and Conventions. We work over a fixed base field k with algebraic closure $\overline{k}$ unless specified otherwise. In this text a k-variety is a geometrically integral separated scheme of finite type over k. A curve is a proper k-variety of dimension one. A surface is a proper k-variety of dimension two.

1. Introduction

A natural question in algebraic geometry is the following one:

Question 1.1. What smooth curves are contained in a given smooth and projective variety X?

For example, a classical result in algebraic geometry is the fact that, when the base field k is infinite, every smooth curve C defined over k can be embedded in $\mathbb{P}^3$. This result is even sharp in the sense that there are strict restrictions on the curves that admit embeddings into $\mathbb{P}^2$; for example, every curve $C \subset \mathbb{P}^2$ has genus $g(C)$ determined by the degree $\deg(C)$ of C under the embedding $C \subset \mathbb{P}^2$ by the degree-genus formula,

$$g(C) = \dim \mathrm{H}^1(C, \mathcal{O}_C) = \frac{1}{2}(\deg(C) - 1)(\deg(C) - 2).$$

This observation partially led the author to the following definition.

Definition 1.2. Let X be a smooth and projective variety. Define $\Sigma_g(X) \subset \mathbb{N} \cup \{0\}$ to be the set of nonnegative integers n for which there exists a smooth curve $C \subset X$ with genus $\mathrm{g}(C) = n$.

In this text, we study the sets $\Sigma_g(X)$ for arbitrary and varying smooth and projective varieties X. This can naturally be seen as a weak version of Question 1.1 since the containment $g(C) \in \Sigma_g(X)$ is an obviously necessary requirement for a variety X to contain a smooth curve C.

In our study, we've been guided by the following three questions. First, for an arbitrary smooth and projective variety X, what does the set $\Sigma_g(X)$ typically look like? Second, which properties of a smooth and projective variety X determine the structure of $\Sigma_g(X)$? Third, what is the coarsest invariant between smooth and projective varieties X and Y that implies $\Sigma_g(X) = \Sigma_g(Y)$? The first two of these three questions are partially answered by the following theorems.

Theorem 1.3. *Let X be a smooth and projective variety with* $\dim(X) \geq 2$. *Then $\Sigma_g(X)$ is an infinite set. Moreover, if one orders the elements of $\Sigma_g(X)$,*

$$\Sigma_g(X) = \{n_1, n_2, n_3, ...\}$$

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with $n_i < n_{i+1}$ for all $i \geq 1$, then there exist constants $\alpha, \beta \geq 0$ and an inequality

$$n_{i+1} \leq \alpha(i+1)^2 + \beta$$

for all sufficiently large $i >> 1$.

Theorem 1.4. *Assume that the base field k is infinite and let X be a smooth and projective variety over k with $\dim(X) \geq 2$. Order the elements of $\Sigma_g(X)$ as an increasing set like*

$$\Sigma_g(X) = \{n_1, n_2, ...\}$$

with $n_{i+1} > n_i$ for all $i \geq 1$.

Suppose that X contains a smooth surface S that admits a nonconstant morphism to a curve C. Then there exist constants $\alpha, \beta \geq 0$ and an inequality

$$n_{i+1} \leq \alpha(i+1) + \beta$$

for all $i \geq 1$.

We establish Theorem 1.3 in Section 2. This theorem essentially characterizes the structure of $\Sigma_g(X)$ for an arbitrary variety X; in particular, $\Sigma_g(X)$ is finite if and only if $\dim(X) \geq 2$ and, the length of gaps between genera of smooth curves on X can grow at most linearly. In Section 2 we also collect a number of examples to illustrate the behavior of $\Sigma_g(X)$ for varying X.

In Section 3, we restrict our attention to the case of surfaces. Here we prove Theorem 1.4 that says: if a smooth and projective variety X contains a smooth surface S that admits a nonconstant morphism to a curve, then the length of gaps between genera of smooth curves on X is bounded by a constant. Unfortunately, our proof relies on implementing Bertini's theorem simultaneously under an infinite collection of embeddings which is why we assume in Theorem 1.4 that the base field k is infinite. We also prove in this section Proposition 3.2, which shows that $\Sigma_g(X)$ has zero density in the set of nonnegative integers for a surface X having Picard rank one, which can be seen as a partial converse to Theorem 1.4.

The last of our questions remains unanswered, even partially, in this note. As an invariant, an obvious observation is that $\Sigma_g(X)$ only depends on the isomorphism class of X, see Remark 2.1. However, it's not clear to what extent this can be refined; we observe in Example 2.3 that there are birational smooth and projective varieties X and Y with differing sets $\Sigma_g(X) \neq \Sigma_g(Y)$. One possibility, that isn't pursued here, is to ask whether the sets $\Sigma_g(X)$ depend only on the $\mathbb{A}^1$-weak equivalence class of X.

2. Properties of $\Sigma_g(X)$

Throughout this section we fix an arbitrary smooth and projective variety X and we write $\Sigma_g(X) \subset \mathbb{N} \cup \{0\}$ for the set of integers that are genera of smooth curves on X (see Definition 1.2).

Remark 2.1. Given another smooth and projective variety Y there is the obvious relation that a closed immersion $X \subset Y$ induces an inclusion $\Sigma_g(X) \subset \Sigma_g(Y)$.

Example 2.2. If $X = \mathrm{Spec}(k)$ then $\Sigma_g(X) = \emptyset$. If X is a curve, then $\Sigma_g(X) = \{g(X)\}$ is just the genus of X.

The following proof shows that the set $\Sigma_g(X)$ is infinite for every such X outside of those considered in Example 2.2.

Proof of Theorem 1.3. Let X be a smooth and projective variety of dimension $\dim(X) \geq 2$ defined over our base field k. By Bertini's theorem, either [Jou83, Théorème 6.10 et Corollaire 6.11] if k is an infinite field or [Poo04, Theorem 1.1 and Proposition 2.7] if k is finite, the variety X contains a smooth and projective variety S of dimension $\dim(S) = 2$. It suffices then by Remark 2.1 to prove the result when $X = S$.

We're going to prove both parts of Theorem 1.3 simultaneously by showing that $\Sigma_g(X)$ contains all values $f(n)$, for all integers n larger than some fixed integer, of a numerical polynomial f of degree 2. To do this, we first remark that, for our surface X and a chosen embedding $X \subset \mathbb{P}^m$, there is an integer n_0 so that for all $n \geq n_0$ one can find a hypersurface $H_n \subset \mathbb{P}^m$ of degree $\deg(H_n) = n$ with intersection $X \cap H_n$ linearly equivalent to a smooth and geometrically integral curve C_n (again this follows from either [Jou83, Théorème 6.10 et Corollaire 6.11] if k is an infinite field or [Poo04, Theorem 1.1 and Proposition 2.7] if k is finite).

For any $n \geq n_0$ we have an exact sequence

$$0 \to \mathcal{O}_X(-n) \to \mathcal{O}_X \to \mathcal{O}_{C_n} \to 0$$

which shows that

$$\chi(C_n, \mathcal{O}_{C_n}) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-n)).$$

Similarly one can get an exact sequence by twisting

$$0 \to \mathcal{O}_X(-n) \to \mathcal{O}_X(n_0 - n) \to \mathcal{O}_{C_{n_0}}(n_0 - n) \to 0.$$

Substitution then shows that

$$\begin{aligned} 1 - g(C_n) = \chi(C_n, \mathcal{O}_{C_n}) &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-n)) \\ &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(n_0 - n)) + \chi(C_{n_0}, \mathcal{O}_{C_{n_0}}(n_0 - n)). \end{aligned}$$

Finally, rearranging the above shows that the genus $g(C_n)$ can be written

$$g(C_n) = \chi(X, \mathcal{O}_X(n_0 - n)) - \chi(C_{n_0}, \mathcal{O}_{C_{n_0}}(n_0 - n)) - \chi(X, \mathcal{O}_X) + 1$$

which is a numerical polynomial in the variable n of degree 2 as claimed. $\square$

In some explicit cases for varieties X one can determine the exact values of $\Sigma_g(X)$.

Example 2.3. If $X = \mathbb{P}^2$, then

$$\Sigma_g(X) = \{0, 1, 3, 6, 10, ...\}$$

is the set of all integers n that can be expressed as $n = \frac{1}{2}(d-1)(d-2)$ for some integer $d \geq 1$ by the degree-genus formula [Har77, Chapter V Example 1.5.1]. (For k finite, one can compare with [Poo04, Section 3.5 Remark]).

If $X = \mathbb{P}^3$ and k is infinite, then

$$\Sigma_g(X) = \{0, 1, 2, ..\} = \mathbb{N} \cup \{0\}$$

since X contains every smooth curve C defined over k.

If $X = \mathbb{P}^1 \times \mathbb{P}^1$ and k is infinite then

$$\Sigma_g(X) = \{0, 1, 2, ...\} = \mathbb{N} \cup \{0\}$$

since X contains curves of arbitrary genus by the bidgree-genus formula (see [Har77, Chapter V Example 1.5.2]). Compared to $Y = \mathbb{P}^2$, this shows that $\Sigma_g(X) \neq \Sigma_g(Y)$ so that this set is not a birational invariant (cf. Corollary 3.4 below).

If $X \subset \mathbb{P}^3$ is a smooth cubic surface defined over an algebraically closed field k, then

$$\Sigma_g(X) = \mathbb{N} \cup \{0\}$$

by [Har77, Chapter V Ex. 4.9].

Example 2.4. Let $X = C_1 \times \cdots \times C_r$ be the product of a finite number of smooth and projective curves $C_1, ..., C_r$. Then for any element $n \in \Sigma_g(X)$ one has $n \geq \min\{g(C_i)\}$. If $g(C_i) = 0$ for some i, then this doesn't say anything so assume $g(C_i) > 0$ for each i. To see this claim, label the projections $\pi_i : X \to C_i$. One can observe that if $D \subset X$ is a smooth and projective curve, then

$\pi_i(D)$ is dominant, hence surjective, for some index $1 \leq i \leq r$. If the base field k has characteristic zero, then the morphism $D \to C_i$ gives an inequality

$$2(g(D)-1) = 2\deg(\pi_i|_D)(g(C_i)-1) + R \geq 2(g(C_i)-1)$$

where the first equality is just the Riemann-Hurwitz formula [Sta19, Tag 0C1D] and the claim follows. If the base field k has characteristic $p > 0$, then the morphism $D \to C_i$ may factor into morphisms [Sta19, Tag 0CD2]

$$D \to D^{(p^n)} \to C_i$$

where $g(D) = g(D^{(p^n)})$ by [Sta19, Tag 0CD0] and where the same Riemann-Hurwitz formula argument now gives $g(D^{(p^n)}) \geq g(C_i)$.

When working over an arbitrary field, there are some arithmetic obstructions to the possible values that can occur among the set $\Sigma_g(X)$. Recall that the index of X is the greatest common divisor of the degrees of the residue fields of closed points on X,

$$\mathrm{ind}(X) = \{\gcd([k(x):k]) : x \in X \text{ is a closed point}\}.$$

If $C \subset X$ is a smooth curve then one has the divisibility relations

$$\mathrm{ind}(X) \mid \mathrm{ind}(C) \mid \deg(\Omega_C) = 2g(C) - 2.$$

Example 2.5. If $X = \mathbf{SB}(A)$ is the Severi-Brauer variety associated to a division algebra A of degree $\deg(A) = 3$, then X is a nontrivial twisted form of $\mathbb{P}^2$ with $\mathrm{ind}(X) = 3$. In this case

$$\Sigma_g(X) = \{1, 10, 28, 55, 91, ...\}$$

is the set of all integers n that can be written $n = \frac{1}{2}(3d-1)(3d-2)$ for some integer $d \geq 1$.

Example 2.6. If $X = \mathbf{SB}(Q_1 \otimes Q_2)$ is the Severi-Brauer variety associated to a product of Quaternion algebras $Q_1 \neq Q_2$ then

$$\Sigma_g(X) = \{1, 3, 5, 7, 9, ...\}$$

is the set of all odd positive integers. To see this, note that since $\mathrm{ind}(X) = 4$ this is as large as the set $\Sigma_g(X)$ can be. To see that all of these values actually do occur, one observes that X contains the variety $Y = \mathbf{SB}(Q_1) \times \mathbf{SB}(Q_2)$ as a twisted Segre subvariety. By applying Bertini's theorem to the very ample classes in

$$\mathrm{Pic}(Y) = 2\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$$

and using the bidgree-genus formula it follows that

$$\Sigma_g(Y) = \{1, 3, 5, ...\}$$

is the set of all integers n that can be written $n = (2d_1 - 1)(2d_2 - 1)$ for any pair of integers $d_1, d_2 \geq 1$.

Example 2.7. This example shows that one can't necessarily deduce the set $\Sigma_g(X)$ from the set $\Sigma_g(X_{\overline{k}})$ of smooth $\overline{k}$-curves lying in $X_{\overline{k}}$ for an algebraic closure $\overline{k} \supset k$.

Let $X = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$ be the Weil restriction. Then $X_{\mathbb{C}} = \mathbb{P}^1 \times \mathbb{P}^1$ so that $\Sigma_g(X_{\mathbb{C}}) = \mathbb{N} \cup \{0\}$ by Example 2.3. But, since $\mathrm{Pic}(X) = \mathbb{Z}$ the set $\Sigma_g(X)$ has zero density in $\mathbb{N} \cup \{0\}$ by Proposition 3.2.

On the other hand, if $Y = \mathbb{P}^1_{\mathbb{R}} \times \mathbf{SB}(Q)$ for the unique nontrivial division algebra Q over $\mathbb{R}$ then $Y_{\mathbb{C}} = \mathbb{P}^1 \times \mathbb{P}^1$ and a similar analysis to Example 2.6 using the inclusion

$$\mathrm{Pic}(Y) = \mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$$

shows that $\Sigma_g(Y) = \mathbb{N} \cup \{0\}$.

In this example, $\mathrm{ind}(X) = 1$ and $\mathrm{ind}(Y) = 2$.

3. Surfaces

Throughout this section we fix a smooth (hence projective) surface S. The main theorem of this section is Theorem 1.4, that shows if X is a smooth and projective variety defined over an infinite field k, containing S, and if S admits a morphism to a curve C then $\Sigma_g(X)$ grows linearly. After giving the proof of this theorem, we deduce some of its consequences. We'll need the following lemma.

Lemma 3.1. *Let S be a smooth and projective surface defined over an infinite field k. Assume that there exists a globally generated line bundle $\mathcal{L} = \mathcal{O}(D)$ on S, corresponding to an effective divisor D on X, that has zero self intersection, i.e. $D^2 = 0$. Then for any very ample smooth curve $C \subset S$ and for any integer $n \geq 1$ there is a very ample smooth curve C_n linearly equivalent to $C + nD$ with genus*

$$g(C_n) = \frac{1}{2}nD(2C + K) + \frac{KC + C^2 + 2}{2}$$

where K is the canonical divisor on S.

Proof. Since D is globally generated, $C + nD$ is very ample for all $n \geq 1$. By Bertini's theorem [Jou83, Théorème 6.10 et Corollaire 6.11] one can find a smooth and geometrically integral curve C_n linearly equivalent to $C + nD$. Applying the adjunction formula to C_n and computing the degree of the canonical bundle on C_n (noting $D^2 = 0$) gives the desired formula for the genus $g(C_n)$. □

Proof of Theorem 1.4. To prove this theorem, we're going to show that $\Sigma_g(X)$ contains the set of integers of the form $an + b$ for all $n \geq 1$ and for some constants a, b (in this case, we won't necessarily find that $a, b \geq 0$ which is why we write a, b here and not α, β).

It suffices by Remark 2.1 to consider only the case $X = S$. We're assuming then that S admits a nonconstant morphism $f : S \to C$ to a curve C and, since C is proper, it suffices to assume $C = \mathbb{P}^1$. In this case, S has a globally generated line bundle $\mathcal{O}(D) = f^*\mathcal{O}(1)$ corresponding to the effective divisor D that is the fiber of f over any rational point of C; in particular $D^2 = 0$.

Thus Lemma 3.1 applies to give a sequence of elements in $\Sigma_g(S)$ depending on an integer $n \geq 1$ that grows linearly in n which concludes the proof. □

As a sort of converse to Theorem 1.4, we give the following proposition (note there are no restrictions on the base field).

Proposition 3.2. *Assume that S is a smooth surface of Picard rank one. Label the elements of*

$$\Sigma_g(S) = \{n_1, n_2, ...\}$$

so that $n_{i+1} > n_i$ for all $i \geq 1$. Then for every pair of constants $\alpha, \beta \geq 0$ there is an index $j_0 = j_0(\alpha, \beta)$ depending on α, β so that

$$n_j \geq \alpha j + \beta$$

for every $j \geq j_0$. Moreover, the set $\Sigma_g(S)$ has zero density in $\mathbb{N} \cup \{0\}$.

Proof. Since S has Picard rank one, any smooth and projective curve $C \subset S$ is linearly equivalent to a multiple $C = nD$ of an ample divisor D. Using the adjunction formula it follows that the genus $g(C)$ equals

$$g(C) = \frac{1}{2}(n^2D^2 + nKD) + 1$$

where K is the canonical class on S. Since D is ample, we have $D^2 > 0$. It follows that even if every positive multiple of D in the Picard group was represented by a smooth curve, then one could still find such an index j_0 depending on the two constants $\alpha, \beta \geq 0$ that has the desired properties.

To see the claim about the density of $\Sigma_g(S)$, we recall that the density $\mu(\Sigma_g(S))$ of $\Sigma_g(S)$ in $\mathbb{N} \cup \{0\}$ is defined as the limit

$$\mu(\Sigma_g(S)) = \lim_{n\to\infty} \frac{\#(\Sigma_g(S) \cap \{0, ..., n\})}{n+1}.$$

The previous paragraph then shows

$$\begin{aligned} 0 \le \mu(\Sigma_g(S)) &= \lim_{n\to\infty} \frac{\#(\Sigma_g(S) \cap \{0, ..., n\})}{n+1} \\ &= \lim_{j\to\infty} \frac{j}{n_j + 1} \\ &\le \lim_{j\to\infty} \frac{j}{(1/2)(j^2D^2 + j(KD)) + 2} = 0 \end{aligned}$$

as desired. □

Example 3.3. Proposition 3.2 applies to $\mathbb{P}^2$, some simple abelian surfaces, and some K3 surfaces among others examples.

Another immediate consequence of Theorem 1.4 is the following corollary that says given a smooth surface S, one can blowup at a subvariety to produce (sometimes many) more curves.

Corollary 3.4. *For any smooth surface S, there is a smooth surface $\tilde{S}$ and a surjection $\pi : \tilde{S} \to S$ satisfying the properties:*

(1) *π restricts to an isomorphism on a dense open $U \subset \tilde{S}$*
(2) *there is a nonconstant morphism $\tilde{S} \to \mathbb{P}^1$.*

In particular if k is infinite, then $\tilde{S}$ satisfies the conditions of Theorem 1.4. □

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