TWISTED HILBERT SCHEMES AND DIVISION ALGEBRAS

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Abstract. Let $\mathscr{X}/S$ be any Severi–Brauer scheme of constant relative dimension n over a Noetherian base scheme S. For each polynomial $\phi(t) \in \mathbb{Q}[t]$, we construct a scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ that fppf locally, on a cover S'/S splitting $\mathscr{X}/S$, is the Hilbert scheme $\mathbf{Hilb}_{\phi(t)}(\mathscr{X}_{S'}/S')$ of the projective bundle $\mathscr{X}_{S'}/S'$.

We then study curves of small degree on a Severi–Brauer variety in order to analyze examples. Our primary interest, in the case that X is the Severi–Brauer variety associated to a central simple k-algebra of degree $n > 2$, over a field k, is the subscheme $\mathrm{Ell}_n(X)$ of $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$ parametrizing curves that are smooth, geometrically connected, and of genus 1.

Contents

1. Introduction

This work originated from the idea that one could study deformations of curves on a Severi–Brauer variety to obtain algebraic information on the structure of the associated central simple algebra. Specifically, this work was an attempt to implement the following program:

(Step 1) for each polynomial $\phi(t)$ of $\mathbb{Q}[t]$, construct a variant of the Hilbert scheme parametrizing closed subschemes of a Severi–Brauer variety X with Hilbert polynomial $\phi(t)$ geometrically;

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(Step 2) for a fixed polynomial $\phi(t) = rt + s$, classify possible curves $C \subset X$ defined over the ground field with Hilbert polynomial $\phi(t)$ over a splitting field for X;

(Step 3) construct a rationally defined deformation of an irreducible curve $C \subset X$ with this Hilbert polynomial to a curve $C^* \subset X$ having the same Hilbert polynomial, and which, additionally, is geometrically a union of rational curves;

(Step 4) study the action of the absolute Galois group on the geometric irreducible components of C^* to obtain specific restrictions on the possible Galois splitting fields of X.

In this paper, we complete (Step 1) in much broader generality and we provide some initial analysis in (Step 2) for specific cases pertaining to curves of minimal degree in a Severi–Brauer variety.

To be precise, we prove in Section 2 (culminating in Theorem 2.5) that for any polynomial $\phi(t) \in \mathbb{Q}[t]$, and for any Severi–Brauer scheme $\mathscr{X}/S$ of relative dimension n over a Noetherian base scheme S, there exists a scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ parametrizing subschemes of $\mathscr{X}$ that are flat and proper over S with Hilbert polynomial $\phi(t)$ over any fppf splitting S'/S for $\mathscr{X}/S$. It turns out that, with only minor changes, one can adapt the proof of representability for the usual Hilbert scheme of a projective bundle to this generalized setting of Severi–Brauer schemes.

We then turn, in Section 3, to the study of those Hilbert schemes $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(X/k)$, for a Severi–Brauer variety X over a fixed field k, that are associated to linear polynomials $\phi(t)$, i.e. to those Hilbert schemes parametrizing subschemes consisting of either curves or curves-and-points. In some cases of minimal degree subschemes, we get very precise information (e.g. in Example 3.11 we give a satisfying picture for the components of $\mathbf{Hilb}^{\mathrm{tw}}_{5t}(X/k)$, for a Severi–Brauer variety X associated to a degree 5 division algebra, that can possibly have a rational point).

Of particular interest, to the author, is the subscheme $\mathrm{Ell}_n(X)$ of $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$ parametrizing smooth and geometrically connected curves of genus one in a Severi–Brauer variety X of dimension $n-1$. It follows from a result of Ein [Ein86] that $\mathrm{Ell}_n(X)$ is geometrically irreducible of dimension n^2 (see Proposition 4.3). For the program outlined above, we also observe that, when X is associated to a division algebra of prime index $n = p > 2$, the only geometrically reducible curves $C \subset X$ appearing in the same component $\mathrm{Ell}_n(X)$ and defined over the ground field are, geometrically, p-gons of lines (Lemma 3.12).

If one could carry out (Step 3) of the above program, then one could use this latter observation to degenerate a smooth curve $C \subset X$ of prime degree p and of genus 1 (when one exists) to a curve which is a geometrically reducible p-gon of lines. This p-gon contains a point of degree p splitting X (the singular point) and the Galois closure of the residue field of this point has Galois group admitting a quotient that is a transitive subgroup of the automorphism group of the p-gon, i.e. either the cyclic group $\mathbb{Z}/p\mathbb{Z}$ or the full dihedral group D_p. With some minimal assumptions this implies that the underlying division algebra is cyclic (i.e. if $p \leq 7$, if the characteristic of k is zero, and if k contains a pth root of unity, see [RS96, Theorem 4.2]).

We conclude, in Section 4, with explicit applications to geometrically elliptic normal curves in Severi–Brauer varieties. We construct generic geometrically elliptic normal curves inside base extensions of generic Severi–Brauer varieties. These generic curves have the property that they specialize to every other geometrically elliptic normal curve inside any other Severi–Brauer variety and we use this observation to compute the periods and indices of these generic curves under an assumption of coprimality to the characteristic of base field (see Theorem 4.7).

Notation. We use the following notation throughout:

- if k is a base field, then we write $\overline{k}$ to denote a fixed algebraic closure of k and k^s to denote the separable closure of k inside $\overline{k}$

Conventions. We use the following conventions throughout:

- a variety is an integral scheme that is separated and of finite type over a base field
- a curve is a proper scheme of pure dimension one that is separated and of finite type over a base field.

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2. Descent for Hilbert Schemes

Let $\mathscr{X}/S$ be a Severi–Brauer scheme of relative dimension n over a Noetherian scheme S. Concretely, this means there exists an fppf cover $S' = \{S_i\}_{i\in I}$ of S and isomorphisms $\mathscr{X}_{S_i} = \mathscr{X} \times_S S_i \cong \mathbb{P}^n_{S_i}$. We call the data consisting of an fppf cover S' and isomorphisms $\epsilon_i : \mathscr{X}_{S_i} \to \mathbb{P}^n_{S_i}$ a *splitting of* $\mathscr{X}/S$. The splitting data $(S_i, \epsilon_i)_{i\in I}$ of $\mathscr{X}/S$ determines a Čech 1-cocycle ξ giving rise to a class in $\check{\mathrm{H}}^1_{fppf}(S, \mathrm{PGL}_{n+1/S})$.

Conversely, descent shows that every element ξ of $\check{\mathrm{H}}^1_{fppf}(S, \mathrm{PGL}_{n+1/S})$ is determined by the splitting data $(S_i, \epsilon_i)_{i\in I}$, for some fppf cover $S' = \{S_i\}_{i\in I}$ of S and some Severi–Brauer scheme $\mathscr{X}/S$ which is uniquely determined by ξ up to isomorphism. For each Čech 1-cocycle ξ one can then choose splitting data $(S_i, \epsilon_i)_{i\in I}$ and, for any polynomial $\phi(t) \in \mathbb{Q}[t]$, descend the Hilbert schemes $\mathbf{Hilb}_{\phi(t)}(\mathbb{P}^n_{S_i}/S_i)$ defined over S_i to a scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ defined over S (cf. Lemma 2.4 below).

The goal for this section is to prove that $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ represents the functor which associates to any locally Noetherian S-scheme T the set of all subschemes of $\mathscr{X}_T$ that are flat and proper over T and which, locally for the fppf cover S'/S, have Hilbert polynomial $\phi(t)$. This is the content of Theorem 2.5; the proof follows the construction of the Hilbert scheme of a projective bundle closely, e.g. as given in [Kol96], making small generalizations so that the same argument can be applied to any Severi–Brauer scheme $\mathscr{X}/S$.

To start, recall from [Qui73, §8.4] that Quillen has constructed a universal vector bundle $\mathcal{J}$ on the Severi–Brauer scheme $\mathscr{X}/S$ having the following property: locally for an fppf cover S'/S splitting $\mathscr{X}/S$, $\mathcal{J}$ admits isomorphisms

$$\mathcal{J}|_{S_i} \cong \mathcal{O}_{\mathbb{P}^n_{S_i}}(-1)^{\oplus n+1} \quad \text{for each } S_i \in S'$$

compatible with the isomorphisms $\mathscr{X}_{S_i} \cong \mathbb{P}^n_{S_i}$ of the splitting. We write $\mathcal{Q} = \mathcal{J}^\vee = \mathcal{H}om(\mathcal{J}, \mathcal{O}_{\mathscr{X}})$ to denote the dual of $\mathcal{J}$ and we call $\mathcal{Q}$ the *Quillen bundle* on the Severi–Brauer scheme $\mathscr{X}/S$.

Lemma 2.1. *Suppose that S is connected and write $\pi : \mathscr{X} \to S$ for the structure map of $\mathscr{X}/S$. Let $\mathcal{F}$ be an S-flat coherent sheaf on $\mathscr{X}$. Then there exists a numerical polynomial $\phi(t) \in \mathbb{Q}[t]$ and an integer N so that the following equality holds*

$$\mathrm{rk}(\pi_*(\mathcal{F} \otimes \mathcal{Q}^{\otimes t})) = \phi(t) \cdot \mathrm{rk}(Q^{\otimes t})$$

for all integers $t \geq N$.

Proof. Let $S' = \{S_i\}_{i\in I}$ be an fppf cover splitting $\mathscr{X}/S$ and write $\pi_i : \mathscr{X}_{S_i} \to S_i$ for map coming from base change. Then, for all $t \geq 1$, there are isomorphisms

$$\pi_*(\mathcal{F} \otimes \mathcal{Q}^{\otimes t})|_{S_i} \cong \pi_{i*}(\mathcal{F}|_{\mathscr{X}_{S_i}} \otimes (\mathcal{O}_{\mathbb{P}^n_{S_i}}(1)^{\oplus n+1})^{\otimes t}) \cong \pi_{i*}(\mathcal{F}|_{\mathscr{X}_{S_i}}(t)^{\oplus (n+1)^t}).$$

Since $\pi_{i*}(\mathcal{F}|_{\mathscr{X}_{S_i}}(t)^{\oplus(n+1)^t}) \cong \pi_{i*}(\mathcal{F}|_{\mathscr{X}_{S_i}}(t))^{\oplus(n+1)^t}$, the $\phi(t)$ of the lemma is necessarily the Hilbert polynomial of $\mathcal{F}|_{\mathscr{X}_{S_i}}$ on $\mathscr{X}_{S_i} \cong \mathbb{P}^n_{S_i}$. □

Definition 2.2. Let $\mathscr{X}/S$ be a Severi–Brauer scheme over a base S. Let $\mathcal{F}$ be an S-flat coherent sheaf on $\mathscr{X}$. For each connected component

$S_\rho \subset S$ we define the *reduced Hilbert polynomial of $\mathcal{F}$ on S_ρ* to be the numerical polynomial $\mathrm{rh}_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ guaranteed to exist by Lemma 2.1. In other words, $\mathrm{rh}_{\mathcal{F}}(t)$ is uniquely characterized by the existence of an integer $N \geq 0$ and equality

$$\mathrm{rk}(\pi_*(\mathcal{F} \otimes \mathcal{Q}^{\otimes t})|_{S_\rho}) = \mathrm{rh}_{\mathcal{F}}(t) \cdot \mathrm{rk}(Q^{\otimes t}) \quad \text{for all } t \geq N.$$

If the reduced Hilbert polynomial of $\mathcal{F}$ on S_ρ is equal to $\mathrm{rh}_{\mathcal{F}}(t)$ for all connected components $S_\rho \subset S$, then we call $\mathrm{rh}_{\mathcal{F}}(t)$ the *reduced Hilbert polynomial of $\mathcal{F}$*. When $\mathcal{F} = \mathcal{O}_V$ is the structure sheaf of a subscheme $V \subset \mathscr{X}$ we write $\mathrm{rh}_V(t)$ instead of $\mathrm{rh}_{\mathcal{O}_V}(t)$.

Remark 2.3. If $\mathscr{X}/S$ is a split Severi–Brauer scheme (i.e. if $\mathscr{X}/S$ is isomorphic over S with a projective bundle $\mathbb{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on S) then, for any S-flat coherent sheaf $\mathcal{F}$ on $\mathscr{X}$, the reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{F}}(t)$ is just the usual Hilbert polynomial $h_{\mathcal{F}}(t)$ with respect to the line bundle $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Lemma 2.4. *Let $\mathscr{X}/S$ be a Severi–Brauer scheme over any scheme S. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{X}$. Then for every polynomial $\phi(t) \in \mathbb{Q}[t]$ there is a locally closed subscheme $S_{\phi(t)} \subset S$ with the property:*

(**f**) *given a morphism $T \to S$, the pullback $\mathcal{F}_T$ on $\mathscr{X}_T$ is flat over T with reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{F}_T}(t) = \phi(t)$ if and only if $T \to S$ factors $T \to S_{\phi(t)} \subset S$.*

Proof. The lemma holds fppf locally over the base S. More precisely, let $S' = \{S_i\}_{i \in I}$ be any fppf cover splitting $\mathscr{X}/S$ with I a finite set and let $\epsilon_i : \mathscr{X}_{S_i} \to \mathbb{P}^n_{S_i}$ be isomorphisms realizing the splitting. Write $T_i = T \times_S S_i$ and $\mathcal{F}_i$ for the pullback of $\mathcal{F}$ to $\mathscr{X}_{T_i}$. Then for each of the indices $i \in I$, there is a locally closed subscheme $S_{i,\phi(t)} \subset S_i$ so that $\mathcal{F}_i$ is flat over T_i with reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{F}_i}(t) = \phi(t)$ if and only if $T_i \to S_i$ factors $T_i \to S_{i,\phi(t)} \subset S_i$. Because of Remark 2.3, the reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{F}_i}(t)$ is just the Hilbert polynomial of $h_{\epsilon_{i*}\mathcal{F}_i}(t)$ and this follows from [Kol96, Theorem I.1.6] which ultimately refers to [Mum66, Lecture 8].

To see that the lemma also holds over S, we note that it's possible to descend the $S_{i,\phi(t)}$ to a scheme $S_{\phi(t)} \subset S$ with $S_{\phi(t)} \times_S S_i = S_{i,\phi(t)}$. Indeed, both of the schemes $S_{i,\phi(t)} \times_S S_j$ and $S_{j,\phi(t)} \times_S S_i$ are uniquely characterized as subschemes of $S_i \times_S S_j$ by the given property with respect to the coherent sheaf $\mathcal{F}_i|_{S_i \times_S S_j} \cong \mathcal{F}_j|_{S_i \times_S S_j}$ on $\mathscr{X}_{S_i \times_S S_j}$. As it's clear that the cocycle condition on any triple product $S_i \times_S S_j \times_S S_k$ is satisfied, it follows from [Sta19, Tag 0247] that $S_{\phi(t)}$ exists as a scheme over S (see also [Sta19, Tag 01OX,Tag 02JR]).

It remains to show that $S_{\phi(t)}$ has property (**f**). Both the flatness of $\mathcal{F}_T$ and the computation for the reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{F}_T}(t)$

can be checked fppf locally for the cover S'/S. The claim follows then from the construction of $S_{\phi(t)}$. $\square$

For any locally Noetherian S-scheme T, write $H^{\phi(t)}_{\mathscr{X}/S}(T)$ for the set

$$H^{\phi(t)}_{\mathscr{X}/S}(T) := \left\{ V \subset \mathscr{X}_T \;\middle|\; \begin{array}{c} V \text{ is proper and flat over } T \\ \text{and } \mathrm{rh}_V(t) = \phi(t) \end{array} \right\}. \tag{1}$$

The association of T to $H^{\phi(t)}_{\mathscr{X}/S}(T)$ defines a contravariant functor from the category of locally Noetherian S-schemes to the category of sets. For a morphism $\rho : T' \to T$, the associated map $H^{\phi(t)}_{\mathscr{X}/S}(T) \to H^{\phi(t)}_{\mathscr{X}/S}(T')$ sends a subscheme $V \subset \mathscr{X}_T$ to $V \times_T T' \subset \mathscr{X}_{T'}$ where the fiber product is taken along the morphism ρ.

Theorem 2.5. *Let $\mathscr{X}/S$ be a Severi–Brauer scheme over a Noetherian base scheme S. Then, for every polynomial $\phi(t) \in \mathbb{Q}[t]$, there exists an S-scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ which represents the functor $H^{\phi(t)}_{\mathscr{X}/S}$ from (1).*

In particular, there is a subscheme

$$\mathbf{Univ}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S) \subset \mathscr{X} \times_S \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$$

and, for any locally Noetherian S-scheme T, there is an equality

$$\mathrm{Hom}_S(T, \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)) = H^{\phi(t)}_{\mathscr{X}/S}(T)$$

where a map $f : T \to \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ corresponds to the subscheme

$$V \cong \mathbf{Univ}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S) \times_{\mathscr{X} \times_S \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)} \mathscr{X} \times_S T.$$

Proof. The proof we give here is, in essence, the same as [Kol96, Proof of Theorem I.1.4]. We're going to break the proof into several steps. First, we construct an S-scheme H together with a scheme $U \subset \mathscr{X} \times_S H$ which end up being $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ and $\mathbf{Univ}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ respectively. Once we've constructed H there will be an obvious functorial map

$$\mathrm{Hom}_S(T, H) \to H^{\phi(t)}_{\mathscr{X}/S}(T) \tag{2}$$

defined as in the theorem statement. The next step will be to construct a map in the other direction

$$H^{\phi(t)}_{\mathscr{X}/S}(T) \to \mathrm{Hom}_S(T, H). \tag{3}$$

The proof will be complete once we show that these two maps are mutually inverse.

Throughout the proof, we'll refer to the following diagram.

$$\begin{tikzcd}
\mathscr{X} \times_S T \arrow[r, dashed, "\tilde{\rho}_{\mathscr{X}}"] \arrow[d, "\pi_T"] \arrow[rr, bend left, "\rho_{\mathscr{X}}"] & \mathscr{X} \times_S \mathscr{Y} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathscr{X} \arrow[d, "\pi"] \\
T \arrow[r, dashed, "\tilde{\rho}"] \arrow[rr, bend right, "\rho"'] & \mathscr{Y} \arrow[r, "\sigma"] & S
\end{tikzcd}$$

For the first part of the proof, we use the following notation:

- $\pi : \mathscr{X} \to S$ is the S-structure map of the Severi–Brauer scheme $\mathscr{X}/S$ of relative dimension n,
- $\phi(t)$ is a fixed polynomial from $\mathbb{Q}[t]$ and $N > 0$ is an integer (chosen to be divisible by $n+1$) so that $h^i(V, \mathcal{O}_V(N)) = 0$ for any subscheme $V \subset \mathbb{P}^n$ with Hilbert polynomial $\phi(t)$ [Kol96, Theorem I.1.5],
- $\mathscr{Y} = \mathbf{Gr}_S(\phi(N), \pi_*\mathcal{L})$ is the Grassmannian S-bundle of rank $\phi(N)$ quotient bundles of the locally free $\pi_*\mathcal{L}$, where $\mathcal{L} = (\det \mathcal{Q})^{\otimes N/(n+1)}$ is the given tensor power of the determinant of the Quillen bundle, with S-structure map $\sigma : \mathscr{Y} \to S$,
- and $p_1, p_2 : \mathscr{X} \times_S \mathscr{Y} \to \mathscr{X}, \mathscr{Y}$ are the first and second projections from the fiber product.

On $\mathscr{Y}$ there is a short exact sequence

$$0 \to \mathcal{U} \to \sigma^*\pi_*\mathcal{L} \to \mathcal{V} \to 0$$

with $\mathcal{V}$ the universal quotient bundle of rank $\phi(N)$ and $\mathcal{U}$ the universal subbundle. Pulling back to $\mathscr{X} \times_S \mathscr{Y}$ we get a map

$$p_2^*\mathcal{U} \to p_2^*\sigma^*\pi_*\mathcal{L} = p_1^*\pi^*\pi_*\mathcal{L} \to p_1^*\mathcal{L} \tag{4}$$

by composing with the (π^*, π_*)-adjunction map. Let $\mathcal{C}$ be the cokernel of this composition. The projection $p_2 : \mathscr{X} \times_S \mathscr{Y} \to \mathscr{Y}$ realizes $\mathscr{X} \times_S \mathscr{Y}$ as a Severi–Brauer scheme over $\mathscr{Y}$ so that we can apply Lemma 2.4 to the sheaf $\mathcal{C} \otimes (p_1^*\mathcal{L}^\vee)$. In this way, we get a subscheme $H \subset \mathscr{Y}$ fitting into a Cartesian diagram

$$\begin{tikzcd}
\mathscr{X} \times_S H \arrow[r, equal] & \mathscr{X} \times_S \mathscr{Y} \times_{\mathscr{Y}} H \arrow[r, "i'"] \arrow[d, "p_2'"] & \mathscr{X} \times_S \mathscr{Y} \arrow[d, "p_2"] \\
 & H \arrow[r, "i"] & \mathscr{Y}
\end{tikzcd}$$

so that $\mathcal{C}' = i'^*(\mathcal{C} \otimes p_1^*\mathcal{L}^\vee)$ is flat over H with reduced Hilbert polynomial $\mathrm{rh}_{\mathcal{C}'}(t) = \phi(t)$. Further, since $\mathcal{C}'$ is a quotient of $\mathcal{O}_{\mathscr{X}\times_S H}$, the sheaf $\mathcal{C}'$ defines a closed subscheme $U \subset \mathscr{X} \times_S H$ with $\mathcal{O}_U = \mathcal{C}'$.

Now let $\rho : T \to S$ be an arbitrary locally Noetherian S-scheme. From the construction of H, any morphism $f : T \to H$ of S-schemes produces an element of $H^{\phi(t)}_{\mathscr{X}/S}(T)$ by pulling back $U \subset \mathscr{X} \times_S H$ along the induced $f_{\mathscr{X}} : \mathscr{X} \times_S T \to \mathscr{X} \times_S H$. This is the definition of (2).

Conversely, from any subscheme $V \subset \mathscr{X} \times_S T$ flat over T with reduced Hilbert polynomial $\mathrm{rh}_V(t) = \phi(t)$ we can identify a morphism $f : T \to H$ as follows. For this part of the proof, we use additionally:

- $\rho_{\mathscr{X}} : \mathscr{X} \times_S T \to \mathscr{X}$ is the first projection from the fiber product $\mathscr{X} \times_S T$ taken with respect to ρ,
- and $\pi_T : \mathscr{X} \times_S T \to T$ is the second projection.

Tensoring the ideal sheaf sequence for V with $\rho^*_{\mathscr{X}}\mathcal{L}$ and pushing forward along π_T gives the exact sequence

$$0 \to \pi_{T*}(\mathcal{I}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L}) \to \pi_{T*}\rho^*_{\mathscr{X}}\mathcal{L} \to \pi_{T*}(\mathcal{O}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L}). \tag{5}$$

The rightmost arrow of this sequence is surjective since the coherent sheaf $R^1\pi_{T*}(\mathcal{I}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L}) = 0$ vanishes; this can be checked after splitting $\mathscr{X}/S$ and using our choice of N, cf. [Kol96, Theorem I.1.5]. Moreover, each of the terms in (5) is locally free by [Har77, Theorem III.12.11] and $\mathrm{rk}(\pi_{T*}(\mathcal{O}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L})) = \phi(N)$.

The base change map

$$\rho^*\pi_*\mathcal{L} \to \pi_{T*}\rho^*_{\mathscr{X}}\mathcal{L}$$

is an isomorphism since it becomes one after an fppf extension of S, see [Nit05, Lemma 5.4]. Hence the surjection $\psi : \rho^*\pi_*\mathcal{L} \to \pi_{T*}(\mathcal{O}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L})$ defines a map

$$\tilde{\rho} : T \to \mathscr{Y} \quad \text{with} \quad \tilde{\rho}^*(\sigma^*\pi_*\mathcal{L} \to \mathcal{V}) = \psi$$

by the functorial description of $\mathscr{Y}$. If we let $\tilde{\rho}_{\mathscr{X}} : \mathscr{X} \times_S T \to \mathscr{X} \times_S \mathscr{Y}$ denote the map obtained by base change, then we find that

$$\begin{aligned} \tilde{\rho}^*_{\mathscr{X}}\mathcal{C} &= \tilde{\rho}^*_{\mathscr{X}}\mathrm{coker}\left(p_2^*\mathcal{U} \to p_2^*\sigma^*\pi_*\mathcal{L} = p_1^*\pi^*\pi_*\mathcal{L} \to p_1^*\mathcal{L}\right) \\ &= \mathrm{coker}\left(\pi_T^*\pi_{T*}(\mathcal{I}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L}) \to \pi_T^*\rho^*\pi_*\mathcal{L} = \rho^*_{\mathscr{X}}\pi^*\pi_*\mathcal{L} \to \rho^*_{\mathscr{X}}\mathcal{L}\right). \end{aligned}$$

The composition factors through the adjunction

$$\pi_T^*\pi_{T*}(\mathcal{I}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L}) \to \mathcal{I}_V \otimes \rho^*_{\mathscr{X}}\mathcal{L},$$

which induces the isomorphism $\tilde{\rho}^*_{\mathscr{X}}(\mathcal{C} \otimes p_1^*\mathcal{L}^\vee) \cong \tilde{\rho}^*_{\mathscr{X}}\mathcal{C} \otimes \rho^*_{\mathscr{X}}\mathcal{L} \cong \mathcal{O}_V$. Since V is flat over T with reduced Hilbert polynomial $\mathrm{rh}_V(t) = \phi(t)$, this implies that $\rho = i \circ f$ factors via a morphism $f : T \to H$ since H satisfies property (**f**). The association sending V to f defines the map in (3). With a moment's thought (and also noting that the factorization above is unique by [Sta19, Tag 01L7]), it's clear the maps (2) and (3) are mutually inverse. This completes the proof. □

Definition 2.6. We'll call $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ *the Hilbert scheme of* $\mathscr{X}/S$ that parameterizes subschemes with reduced Hilbert polynomial $\phi(t)$. The superscript tw is a reminder that this is a twist of one of the usual Hilbert schemes of a projective bundle as the next remark notes.

Remark 2.7. If $\mathscr{X}/S$ is split, i.e. if $\mathscr{X}/S$ is a projective bundle $\mathbb{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on S, then the above theorem recovers the usual Hilbert scheme $\mathbf{Hilb}_{\phi(t)}(\mathbb{P}_S(\mathcal{E})/S)$. This also shows the following statement: if $\mathscr{X}/S$ is any Severi–Brauer scheme over a Noetherian base scheme S, and if S'/S is an fppf cover splitting $\mathscr{X}/S$, then there are splitting isomorphisms

$$\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)\times_S S'\cong \mathbf{Hilb}_{\phi(t)}(\mathscr{X}_{S'}/S')$$

as claimed in the beginning of this section. Consequently, the scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ inherits any property of $\mathbf{Hilb}_{\phi(t)}(\mathscr{X}_{S'}/S')$ that can be checked fppf locally on the base, i.e. being finite-type, proper, or smooth over S holds if it also does over S'.

Remark 2.8. Given any Severi–Brauer scheme $\mathscr{X}/S$ with structure map $\pi:\mathscr{X}\to S$, it follows from [Sta19, Tag 01VR] that $\mathcal{L}=\det\mathcal{Q}$ is a π-relatively very ample line bundle. Hence π is projective with respect to $\mathcal{L}$ and for any polynomial $\phi(t)\in\mathbb{Q}[t]$ there is a usual Hilbert scheme $\mathbf{Hilb}_{\phi(t)}(\mathscr{X}/S)$ parametrizing flat and proper subschemes of $\mathscr{X}$ whose Hilbert polynomial with respect to $\mathcal{L}$ is $\phi(t)$. If $\mathscr{X}$ has constant relative dimension $n-1$ over S, then there is an isomorphism

$$\mathbf{Hilb}^{tw}_{\phi(t)}(\mathscr{X}/S)\cong \mathbf{Hilb}_{\phi(nt)}(\mathscr{X}/S).$$

The primary benefit in working with $\mathbf{Hilb}^{tw}_{\phi(t)}(\mathscr{X}/S)$ instead of the usual Hilbert scheme is in the canonical nature of its construction (e.g. the twisted and usual Hilbert scheme are both realized as subschemes of certain projective bundles; however, in these embeddings, the relative codimension of the twisted Hilbert scheme will always be much lower than that of the usual one).

The infinitesimal theory of $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ can also be checked on an fppf cover of the base, so we get the following corollary using the fact that the scheme $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ is fppf locally, e.g. on a cover S'/S splitting $\mathscr{X}/S$, isomorphic to $\mathbf{Hilb}_{\phi(t)}(\mathbb{P}^n_{S'}/S')$.

Corollary 2.9. *Let $\mathscr{X}/S$ be a Severi–Brauer scheme over S. Let $s\in S$ be a point, let F be a field, and let $p:\mathrm{Spec}(F)\to s$ be a morphism.*

Let $V \subset \mathscr{X}_F$ be a subscheme with ideal sheaf $\mathcal{I}_V$ and reduced Hilbert polynomial $\mathrm{rh}_V(t) = \phi(t)$*. Then the following are true:*

(1) The Zariski tangent space of $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}_F/F)$ at the F-point given by V via Theorem 2.5 is naturally isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{\mathscr{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V).$$

(2) The dimension of every irreducible component of $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}_F/F)$ at the F-point defined by V is at least

$$\dim_F \mathrm{Hom}_{\mathcal{O}_{\mathscr{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) - \dim_F \mathrm{Ext}^1_{\mathcal{O}_{\mathscr{X}_F}}(\mathcal{I}_V, \mathcal{O}_V) + \dim_s S.$$

(3) If $V \subset \mathscr{X}_F$ is (fppf) locally unobstructed, then the dimension of every irreducible component of $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ at any point in the image of the point defined by V is at least

$$\dim_F \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V) - \dim_F \mathrm{H}^1(V, \mathcal{H}om(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)) + \dim_s S.$$

Moreover, in either of the cases (2) or (3) above, if the lower bound given for the dimension is equal to the dimension of every irreducible component of $\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S)$ at the point defined by V, then the map

$$\mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(\mathscr{X}/S) \to S$$

is a local complete intersection morphism at that point.

Proof. This is a combination of [Kol96, Theorems I.2.10 and I.2.15]. See [Kol96, Definition I.2.11] for the definition of locally unobstructed subschemes. ☐

3. Classifying subschemes

From now on, we work in the following setting: we fix a base field k, a k-central simple k-algebra A, and we let $X = \mathbf{SB}(A)$ be the associated Severi–Brauer variety of A. We use the triple (d, n, m) to refer to the degree, index, and exponent of A respectively, i.e.

$$d = \deg(A), \quad n = \mathrm{ind}(A), \quad m = \exp(A).$$

In this section, we analyze the subschemes of X corresponding to points in the Hilbert scheme $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$. We assume throughout this section that $d > 2$ so that X is not a curve itself.

Lemma 3.1. *Let $C \subset X$ be a curve. Let p be a prime number. Then the degree* $\deg(C)$ *satisfies the following*

$$v_p(\deg(C)) \geq \begin{cases} v_p(n) & \text{if } p \text{ is odd} \\ v_p(n) - 1 & \text{if } p = 2. \end{cases}$$

In other words, the integer n divides $\deg(C)$ *if n is odd and the integer $n/2$ divides* $\deg(C)$ *if n is even.*

Proof. Let F be any splitting field for X. The degree $\deg(C)$ is defined as the unique integer so that there is an equality $[C_F] = \deg(C)[L]$ inside the Chow group $\mathrm{CH}_1(X_F)$ where $L \subset X_F \cong \mathbb{P}_F^{d-1}$ is any line. The degree $\deg(C)$ is independent of the choice of splitting field F.

To check the given divisibility relations, it suffices to work p-locally in the group $\mathrm{CH}_1(X_F) \otimes \mathbb{Z}_{(p)}$. By a corestriction-restriction argument, it therefore suffices to assume $n = p^r$ is a prime power of p. The case that p is odd is [Mac21, Theorem 4.7]. So we can assume that $p = 2$.

By first replacing k with some (possibly large) field extension that doesn't change the index of A, we can assume that $m = \exp(A) = 2$. Let $e \in \mathrm{CH}^1(X)$ be the class of a divisor that has degree $m = \exp(A)$ over any splitting field for X. Let p be any closed point of X with $[k(p) : k] = n = 2^r$ (the residue field of such a point corresponds to a maximal subfield of the underlying division algebra and the point itself corresponds to a minimal left ideal inside the algebra over this field). Set $F = k(p)$ and let $q \in X_F$ be any F-rational point inside p_F. Then since $\mathrm{CH}_0(X) = \mathbb{Z}$ is generated by the class of the point p, we find (by restricting to F the relation $e \cdot [C] = a[p]$ that holds over k for some integer $a \geq 1$) that

$$e_F \cdot [C_F] = 2\deg(C)[q] = a[p_F] = a2^r[q].$$

In other words, we find 2^{r-1} divides the degree $\deg(C)$. □

Remark 3.2. In fact, when the index n is a power of 2, it follows that for any field F splitting X, the image of $\mathrm{CH}_1(X)$ inside $\mathrm{CH}_1(X_F) = \mathbb{Z}$ by restriction is exactly $(n/2)\mathbb{Z}$. Generators of the image are exactly the restrictions of the classes $c_{n-2}(\zeta_X(1))c_n(\zeta_X(1))^{(d/n)-1}$ and $c_1(\zeta_X(m))^{d-2}$ constructed in [KM19, Appendix A].

Example 3.3. In this example, we construct some curves in Severi–Brauer varieties associated to central simple k-algebras of index $n = 2^r$ with 2-adic valuation of the degree strictly smaller than r.

For an example of a curve C with minimal possible degree in X, let $A = Q_1 \otimes Q_2$ be a biquaternion algebra of index 4 split by a biquadratic extension $F = k(\sqrt{a}, \sqrt{b})$ with Galois group $\mathrm{Gal}(F/k) = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Let p be a point on X with residue field $k(p) = F$ and identify the points in p_F with elements of $\mathrm{Gal}(F/k)$ so that the action of G on p_F is the canonical one. In $X_F \cong \mathbb{P}_F^3$ let L_1 be the line passing through the points $(0,0)$ and $(0,1)$ and let L_2 be the line passing through the points $(1,0)$ and $(1,1)$. Then $L_1 \cup L_2$ forms a Galois orbit, hence it descends to the ground field k to give a curve $C \subset X$ with $\deg(C) = 2$.

If the division algebra $A = Q_1 \otimes \cdots \otimes Q_r$ is an r-fold ($r > 2$) product of distinct quaternion algebras split by a multi-quadratic extension F/k with $\mathrm{Gal}(F/k) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r}$, then there is a point p on X with residue field F. One can pass a Galois orbit of lines through p_F which is essentially an r-dimensional cube $I_r = [0,1]^r$ with lines replacing the edges of the cube. The curve $C \subset X$ that one gets from this cube has degree $\deg(C) = r2^{r-1}$ (equal to the number of edges of the r-cube). If $r = 3$ then this curve has arithmetic genus $h^1(C, \mathcal{O}_C) = 2^{3-2}\binom{3}{2} - 1$ (equal to the number of faces of the 3-cube minus 1).

In general, any division algebra A of index $n = 2^r$ is split by a separable field extension F/k of degree $[F : k] = 2^r$. For this field F, there is a point p on X with residue field $k(p) = F$ and p_{k^s} contains 2^r points over a separable closure k^s of k. Passing a line between every pair of points in p_{k^s} gives a $\mathrm{Gal}(k^s/k)$ orbit that descends to a curve $C \subset X$. The degree of C is equal to the number of edges in the complete graph K_n, i.e. $\deg(C) = \binom{n}{2} = 2^{r-1}(2^r - 1)$.

Lemma 3.4. *Assume that A is a division k-algebra, i.e. assume $n = d$. Let $V \subset X$ be any subscheme of X containing an irreducible component C of dimension $\dim(C) \geq 1$. Then V is geometrically nondegenerate.*

Furthermore, if $C \subset X$ is any geometrically integral curve in X with degree $\deg(C) = p$ for some integer $p \geq 1$, then the geometric genus of C is bounded above by

$$g_{geom}(C) \leq (d-2)\frac{q(q-1)}{2} + qr$$

where q, r are the quotient and remainder of dividing $p - 1$ by $d - 2$, i.e. where $p - 1 = q(d-2) + r$ and $0 \leq r < d - 2$.

Proof. Let $\overline{k}$ be an algebraic closure of k and k^s a separable closure of k inside $\overline{k}$. To see that $V_{\overline{k}}$ is nondegenerate in $X_{\overline{k}} \cong \mathbb{P}^{d-1}_{\overline{k}}$, it suffices to show that $C_{\overline{k}}$ is contained in no hyperplane.

Assume to the contrary that there is a hyperplane H' containing $C_{\overline{k}}$, and let H be the unique hyperplane inside X_{k^s} with $H \times_{k^s} \overline{k} = H'$. Let α be a 1-cocycle of $G = \mathrm{Gal}(k^s/k)$ representing X in $\mathrm{H}^1(G, \mathrm{PGL}_n(k^s))$. Then C_{k^s} is contained in each of the (finitely many) Galois orbits of H coming from α. In particular, we get an inclusion

$$C_{k^s} \subset \bigcap_{g \in G} gH$$

with the right hand side a Galois invariant linear subspace of X_{k^s}. Since this subspace would necessarily descend to the field k, this contradicts the fact that A was assumed to be a division k-algebra (which implies that X has no twisted linear subvarieties).

Now suppose that C is a geometrically integral curve in X. To bound the geometric genus of C, it suffices to work over the algebraic closure. In particular, we can assume that C is an integral curve nondegenerate inside of $\mathbb{P}^{d-1}$. Then the claim is just Castelnuovo's bound [Har81]. □

Remark 3.5. The above proof even shows the more general statement that, if A is any central simple k-algebra and if $V \subset X$ is any subscheme of X containing an irreducible component C of dimension $\dim(C) \geq 1$, then $V_{\overline{k}}$ is set-theoretically contained in a hyperplane of $X_{\overline{k}}$ if and only if V is set-theoretically contained in a twisted linear subvariety $Y \subsetneq X$.

In particular, if A is a division k-algebra, then both $V_{\overline{k}}$ and $(V_{\overline{k}})_{red}$ are geometrically nondegenerate.

Remark 3.6. Assume that A is a k-division algebra so that $d = n$. Then for any geometrically integral curve $C \subset X$ with $\deg(C) \leq n$, we find $g_{geom}(C) \leq 1$ by applying Lemma 3.4 (and if $\deg(C) < n$ then also $g_{geom}(C) < 1$).

Suppose $C \subset X$ as above has both $\deg(C) \leq n$ and $g_{geom}(C) = 0$. Then, in this case, since C is geometrically integral, the normalization C^ν of C is smooth, geometrically connected, and geometrically rational. So there is a point p on C^ν with $[k(p) : k] = 2$. As C^ν maps to X, this can only happen if $d = 2$.

It follows that any smooth and geometrically connected curve $C \subset X$ with $\deg(C) \leq n$ has genus $g_{geom}(C) = h^1(C, \mathcal{O}_C) = 1$ and $\deg(C) = n$.

Typically the arithmetic genus $h^1(C, \mathcal{O}_C)$ gives more information about a curve C and its embedding $C \subset X$. The next two lemmas give technical tools that can allow one to determine the possible values for the arithmetic genera of curves in X in some cases.

Lemma 3.7. *Let p be a prime number. Suppose that $V \subset X$ is any subscheme whose irreducible components $P_i \subset V$ have $\dim(P_i) \leq 1$. Then we have*

$$v_p(h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V)) \geq \begin{cases} v_p(n) & \text{if } p \text{ is odd} \\ v_p(n) - 1 & \text{if } p = 2. \end{cases}$$

In other words, the integer n divides $\chi(V, \mathcal{O}_V)$ if n is odd and the integer $n/2$ divides $\chi(V, \mathcal{O}_V)$ if n is even.

Proof. It follows from flat base change [Sta19, Tag 02KH] that this can be checked geometrically. This fits into a commutative diagram

$$\begin{tikzcd} K(X_{\overline{k}}) \arrow[r] & \mathbb{Z} \\ K(X) \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, equal] \end{tikzcd}$$

where the horizontal arrows are pushforwards along the structure map (with the identification $K(\mathrm{Spec}(k)) = \mathbb{Z} = K(\mathrm{Spec}(\overline{k}))$ of Grothendieck rings) and the vertical arrows are induced by extension of scalars. The class $[\mathcal{O}_V]$ in $K(X)$ sits in the topological filtration $\tau_1(X) \subset K(X)$ generated by coherent sheaves supported in dimension 1 or less.

The image of $\tau_1(X)$ under the left vertical map is given in [Mac21, Theorem 4.7] under the assumptions that p is odd and $n = p^r$. In that particular case, $\tau_1(X) = p^r\tau_1(X_{\overline{k}})$ with generators $p^r[\mathcal{O}_{\mathbb{P}^1}]$ and $p^r[\mathcal{O}_q]$ for the class of a $\overline{k}$-rational point $q \in X_{\overline{k}}$. The horizontal arrows in the diagram above take the class of a coherent sheaf $[\mathcal{F}]$ to $\chi(X, \mathcal{F})$. Writing $[\mathcal{O}_V] = ap^r[\mathcal{O}_{\mathbb{P}^1}] + bp^r[\mathcal{O}_q]$ it follows that

$$p^r(a + b) = \chi(X, \mathcal{O}_V) = \chi(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V).$$

Since it suffices to check the claim with coefficients in $\mathbb{Z}_{(p)}$, the case that p is odd follows from the particular case above by a restriction-corestriction argument. The case $p = 2$ follows by a similar argument using Lemma 3.1 to show $\tau_1(X) \subset 2^{r-1}\tau_1(X_{\overline{k}})$ when $n = 2^r$. □

Lemma 3.8. *Suppose that $V \subset X$ is any subscheme whose irreducible components $P_i \subset V$ have $\dim(P_i) \leq 1$. Assume that $\mathrm{rh}_V(t) = rt + s$ for some integers r, s with $r \geq 1$. Then $h^1(V, \mathcal{O}_V) \leq \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$.*

Proof. This is a bit overkill but, since we have

$$\mathbf{Hilb}^{\mathrm{tw}}_{rt+s}(X/k) \times_k \overline{k} \cong \mathbf{Hilb}_{rt+s}(\mathbb{P}^{d-1}_{\overline{k}}/\overline{k})$$

it's enough to show that the right hand side is empty whenever there is an inequality $h^1(V, \mathcal{O}_V) > \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$. This is proved in [Har66, Corollary 5.7]. More specifically, Hartshorne shows there that $\mathbf{Hilb}_{rt+s}(\mathbb{P}^{d-1}_{\overline{k}}/\overline{k})$ is nonempty if and only if one has $m_0 \geq m_1 \geq 0$ when $rt + s$ is written as

$$\begin{aligned} rt + s &= \binom{t}{1} - \binom{t - m_0}{1} + \binom{t+1}{2} - \binom{t + 1 - m_1}{2} \\ &= m_0 + m_1 t + \frac{1}{2}(m_1 - m_1^2). \end{aligned}$$

Comparing coefficients in the above gives

$$r = m_1 \quad \text{and} \quad s = \chi(V_{\overline{k}}, \mathcal{O}_{V_{\overline{k}}}) = \chi(V, \mathcal{O}_V) = m_0 + \frac{1}{2}(m_1 - m_1^2).$$

Equivalently, since $\chi(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) - h^1(V, \mathcal{O}_V)$ this implies

$$h^0(V, \mathcal{O}_V) - s = h^1(V, \mathcal{O}_V) = \frac{1}{2}(m_1^2 - m_1) - m_0 + h^0(V, \mathcal{O}_V).$$

Now $\mathbf{Hilb}_{rt+s}(\mathbb{P}^{d-1}_{\overline{k}}/\overline{k})$ is nonempty if and only if $m_0 \geq m_1 \geq 0$ if and only if $r \geq 0$ and $0 \leq h^1(V, \mathcal{O}_V) \leq \frac{1}{2}(r^2 - 3r) + h^0(V, \mathcal{O}_V)$. $\square$

Proposition 3.9. *Suppose that A is a division k-algebra with index n. Let $V \subset X$ be any subscheme with $\mathrm{rh}_V(t) = f(n)t + s$ where $f(n) = n$ if n is odd and $f(n) = n/2$ if n is even. Then the following are true.*

(1) There is a unique irreducible component $C \subset V$ with $\dim(C) = 1$. Moreover, C is generically reduced and $\deg(C) = f(n)$.
(2) If the index $n = p$ is prime, then the curve $C \subset V$ is geometrically generically reduced.
(3) If $s = 0$ and if the index $n = p$ is prime, then the curve $C \subset V$ is also geometrically connected.

Proof. As $\mathrm{rh}_V(t) = f(n)t + s$ has degree $\deg(\mathrm{rh}_V(t)) = 1$, and since $\mathrm{rh}_V(t)$ is geometrically the Hilbert polynomial of $V_{\overline{k}}$, the dimension of any irreducible component P_i of V satisfies $\dim(P_i) \leq 1$. If there were multiple components P_i of $\dim(P_i) = 1$, then it would follow that $\deg(P_i) < f(n)$ which is impossible by Lemma 3.1.

Similarly, if the unique irreducible component $C \subset V$ of dimension $\dim(C) = 1$ had $\mathrm{length}_{k(C)}\mathcal{O}_{C,\eta} > 1$ at the generic point η of C, then we would find $\deg(C_{red}) < \deg(C)$ which also contradicts Lemma 3.1. Hence the curve $C \subset V$ is also generically reduced, proving (1).

To prove (2), i.e. to show that C is geometrically generically reduced when $n = p$ is prime, we can assume $n = p > 2$. If C is geometrically irreducible, then geometrically we have

$$\deg(C) = m_{C_{\overline{k}}} \deg(C_{\overline{k}}) \quad \text{with} \quad m_{C_{\overline{k}}} = \mathrm{length}_{\overline{k}(C_{\overline{k}})} \mathcal{O}_{C_{\overline{k}}, \overline{\eta}}$$

where $\overline{\eta}$ is the generic point of $C_{\overline{k}}$. If $m_{C_{\overline{k}}} = p$, then $(C_{\overline{k}})_{red}$ is a line in $X_{\overline{k}} \cong \mathbb{P}^{n-1}_{\overline{k}}$, hence topologically degenerate, contradicting Remark 3.5. So we must have $m_{C_{\overline{k}}} = 1$ in which case $C_{\overline{k}}$ is generically reduced.

On the other hand, if C is geometrically reducible, then the Galois group $G = \mathrm{Gal}(k^s/k)$ acts transitively on the irreducible components of $C_{\overline{k}}$ and all of these irreducible components have the same degree. Since there are at least two irreducible components of $C_{\overline{k}}$, there must be exactly p, say $C_1, ..., C_p$, each with degree $\deg(C_i) = 1$. Hence C is also geometrically generically reduced in this case.

Suppose now that $s = 0$ and $n = p > 2$ is a prime number. For (3), we suppose that the unique curve $C \subset V$ is geometrically disconnected and aim for a contradiction. Since the degree of C is $\deg(C) = p$, there are exactly p connected components $C_1, ..., C_p$ of $C_{\overline{k}}$ with $\deg(C_i) = 1$. Hence $(C_i)_{red} \cong \mathbb{P}^1_{\overline{k}}$. Considering the ideal sheaf sequence

$$0 \to \mathcal{N} \to \mathcal{O}_{C_{\overline{k}}} \to \mathcal{O}_{(C_{\overline{k}})_{red}} \to 0,$$

we find that $\mathrm{Supp}(\mathcal{N})$ is a finite set of closed points. Therefore

$$h^1(C, \mathcal{O}_C) = h^1(C_{\overline{k}}, \mathcal{O}_{C_{\overline{k}}}) = h^1((C_{\overline{k}})_{red}, \mathcal{O}_{(C_{\overline{k}})_{red}}) = 0.$$

But $h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V) \neq 0$ by the assumption that the reduced Hilbert polynomial of V is $\mathrm{rh}_V(t) = pt$. Hence if $n = p > 2$ is prime, then C is geometrically connected. □

Remark 3.10. Proposition 3.9 implies that, if A is a division k-algebra of prime index $n = p$, then any reduced curve $C \subset X$ with $\deg(C) = p$ is geometrically reduced [Sta19, Tag 04KS].

Example 3.11. If A is a division k-algebra with index $n = 5$, then we can say something about possible subschemes $V \subset X$ with $\mathrm{rh}_V(t) = 5t$. Let $C \subset V$ be the unique curve sitting in V with degree $\deg(C) = 5$. Then C is geometrically connected (by Proposition 3.9) and C is either reduced or nonreduced. If C is reduced, then C is geometrically reduced (by Remark 3.10). In this case $h^0(C, \mathcal{O}_C) = 1$ so that Lemma 3.8 implies $h^1(C, \mathcal{O}_C) \leq \frac{1}{2}(25 - 15) + 1 = 6$. With Lemma 3.7 we get that 5 divides $1 - h^1(C, \mathcal{O}_C)$. So either $h^1(C, \mathcal{O}_C) = 1$ or $h^1(C, \mathcal{O}_C) = 6$.

If C is reduced and geometrically reducible, then $C_{\overline{k}}$ is the union of 5 irreducible components $C_1, ..., C_5$ each with $\deg(C_i) = 1$. The singular points of $C_{\overline{k}}$ form a Galois orbit, so that they span $X_{\overline{k}} \cong \mathbb{P}^4_{\overline{k}}$ linearly. In particular, $C_{\overline{k}}$ is a union of 5 lines passing through at least 5 points. We show in Lemma 3.12 below that $C_{\overline{k}}$ is essentially a 5-gon of lines with $h^1(C, \mathcal{O}_C) = 1$. Since $\chi(C, \mathcal{O}_C) = 0$, this implies $C = V$.

Otherwise C is geometrically integral and the normalization of C is a smooth genus 1 curve by Remark 3.6. Then either C is smooth and $h^1(C, \mathcal{O}_C) = 1$, or $h^1(C, \mathcal{O}_C) = 6$ and C is singular [Sta19, Tag 0CE4]. In the former case we again find $V = C$ and, in the latter case, we find $V = C \cup p$ for some Artinian subscheme $p \subset X$ with $h^0(p, \mathcal{O}_p) = 5$. But, since 5 divides the degree of any closed point of X, we find that p is a closed point with $[k(p) : k] = 5$.

When C is nonreduced, we consider instead the reduced subscheme $C_{red} \subset C$ which still has $\deg(C_{red}) = 5$ since C is generically reduced.

The scheme C_{red} is both geometrically connected (since this is true for C by Proposition 3.9) and geometrically reduced (from Remark 3.10). Hence $h^0(C_{red}, \mathcal{O}_{C_{red}}) = 1$ and, similar to the reduced case, we find that either $h^1(C_{red}, \mathcal{O}_{C_{red}}) = 1$ or $h^1(C_{red}, \mathcal{O}_{C_{red}}) = 6$ from both Lemma 3.8 and Lemma 3.7.

Now from the ideal sheaf sequence for $C_{red} \subset C$ we find (in)equalities

$$h^0(C, \mathcal{O}_C) > h^0(C_{red}, \mathcal{O}_{C_{red}}) \quad \text{and} \quad h^1(C, \mathcal{O}_C) = h^1(C_{red}, \mathcal{O}_{C_{red}}).$$

It follows that if we write $V = C \sqcup p$ as the disjoint union of C and a possibly empty Artinian scheme p, then

$$0 = \chi(V, \mathcal{O}_V) = \chi(C, \mathcal{O}_C) + r > \chi(C_{red}, \mathcal{O}_{C_{red}}) + r$$

with $r = h^0(p, \mathcal{O}_p) \geq 0$. So $\chi(C_{red}, \mathcal{O}_{C_{red}}) < 0$ and $h^1(C_{red}, \mathcal{O}_{C_{red}}) = 6$. But then $h^0(V, \mathcal{O}_V) = 6$ as well since we have $h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V)$. Since there are no closed points on X of degree less than 5, it follows that $V = C$ and $h^0(C, \mathcal{O}_C) = 6$.

Lemma 3.12. *Let A be a division k-algebra of prime index $n = p > 2$. Let $V \subset X$ be any subscheme with $\mathrm{rh}_V(t) = pt$ and let $C \subset V$ be the unique curve from Proposition 3.9. If C is geometrically reducible, then $C_{\overline{k}}$ is a p-gon of lines through p points spanning $X_{\overline{k}} \cong \mathbb{P}^{p-1}_{\overline{k}}$.*

Proof. Since C is geometrically reducible and of degree $\deg(C) = p$, we know that $(C_{red})_{\overline{k}}$ is the union of p-lines $L_1, ..., L_p$. The Galois group $G = \mathrm{Gal}(k^s/k)$ acts transitively on the set of these lines $\{L_1, ..., L_p\}$, giving a map $G \to S_p$ whose image contains a p-cycle. Proposition 3.9 shows that the curve C is geometrically connected, so we can find an element $g \in G$ so that $L_1 \cap gL_1 \neq \emptyset$ and, after possibly relabeling the lines $L_1, ..., L_p$ we can assume

(1) $L_k = g^{k-1}L_1$ for all $1 \leq k \leq p$,
(2) $L_i \cap L_{i+1} \neq \emptyset$ for all $i = 1, ..., p-1$
(3) and $L_p \cap L_1 \neq \emptyset$ also.

Assume that there is a hyperplane $H \subset X_{\overline{k}} \cong \mathbb{P}^{p-1}_{\overline{k}}$ containing all lines L_i with $i \neq p$. Then, in particular, this H contains the set of all singular points of $C_{\overline{k}}$ which are the union of Galois orbits under G. Since H doesn't contain L_p, we have that gH doesn't contain $gL_p = L_1$. The intersection of these translates is then empty by a dimension count,

$$\bigcap_{g \in G} gH = \emptyset.$$

But this would imply C is smooth, a contradiction since C is assumed singular. Hence no H can contain $L_1, ..., L_{p-1}$.

Now we go inductively. Starting with L_1, we add L_2 with $L_1 \cap L_2 \neq \emptyset$ by our choice of labeling. We get that $L_1 \cup L_2$ is contained in a unique linear subspace H_1 with $\dim(H_1) = 2$. Now we consider L_3. Adding L_3 to $L_1 \cup L_2$, we know that $L_2 \cap L_3 \neq \emptyset$ so that $L_1 \cup L_2 \cup L_3$ is contained in some linear subspace H_2 with $\dim(H_2) = 3$. We also know $L_1 \cup L_2 \cup L_3$ is not contained in H_1 since, if it were, then $L_1 \cup \cdots \cup L_{p-1}$ would be contained in a hyperplane. Hence $\#(L_3 \cap H_1) = 1$ and the subspace H_2 is uniquely determined as well.

Repeating this process, we find for all $1 \leq i < p-1$ linear subspaces H_i of dimension $\dim(H_i) = i+1$ so that each H_i contains $L_1 \cup \cdots \cup L_{i+1}$, and $\#(L_{i+1} \cap H_{i-1}) = 1$. Finally, we know $\#(L_p \cap (L_1 \cup \cdots \cup L_{p-1})) \geq 2$ (since otherwise there is a unique singular point) and we claim that actually equality holds. If this inequality were strict, then $L_p \cap L_i \neq \emptyset$ for some $i \neq 1, p-1$. But then

$$gL_p \cap gL_i = L_1 \cap L_{i+1} \neq \emptyset$$

and $1 < i+1 \leq p-1$, contradicting that $\#(H_{i-1} \cap L_{i+1}) = 1$.

We think of $(C_{red})_{\overline{k}}$ as a graph with singular points as vertices and lines as edges; the above shows that the graph associated to $(C_{red})_{\overline{k}}$ is a p-gon on exactly p-vertices. These p-vertices form a Galois orbit under G, so they must span $X_{\overline{k}}$. Corollary 3.13 below shows that $C = C_{red}$, which completes the proof. $\square$

Corollary 3.13. *Let A be a division k-algebra of prime index $n = p$. Let $V \subset X$ be any subscheme with $\mathrm{rh}_V(t) = pt$ and let $C \subset V$ be the unique curve found in Proposition 3.9. If C is geometrically reducible, then C is reduced, $h^1(C, \mathcal{O}_C) = 1$, and $V = C$.*

Proof. The proof of Lemma 3.12 describes how to construct $(C_{red})_{\overline{k}}$ as a union of lines. One can use this construction to compute $h^1(C, \mathcal{O}_C) = 1$ from the exact sequence (7) below and the observation that

$$h^1(C, \mathcal{O}_C) = h^1(C_{red}, \mathcal{O}_{C_{red}}) = h^1((C_{red})_{\overline{k}}, \mathcal{O}_{(C_{red})_{\overline{k}}})$$

where the first equality comes from C being generically reduced and the second from flat base change.

Then $1 = h^1(C, \mathcal{O}_C) = h^1(V, \mathcal{O}_V) = h^0(V, \mathcal{O}_V)$ as $\mathrm{rh}_V(t) = pt$. Since

$$0 \neq H^0(C, \mathcal{O}_C) \subset H^0(V, \mathcal{O}_V),$$

we find that $h^0(C, \mathcal{O}_C) = 1$. Hence C is reduced and $V = C$. $\square$

4. Generic geometrically elliptic normal curves

It can be difficult to say anything complete regarding the schemes $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$, as done in Example 3.11, for general X. However, we can still analyze specific irreducible components of $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$, in some particular cases of Severi–Brauer variety X, to some benefit.

From now on we write

$$\psi_X : \mathbf{Univ}^{\mathrm{tw}}_{\phi(t)}(X/k) \to \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(X/k) \tag{6}$$

for the canonical map coming from the projection. (By slight abuse of notation, we use the same ψ_X regardless of the function $\phi(t)$ under consideration). For each irreducible component $V \subset \mathbf{Hilb}^{\mathrm{tw}}_{\phi(t)}(X/k)$ we let η_V denote the generic point of V. If $\phi(t) = rt + s$ is linear then, for each such V, the generic fiber $\psi_X^{-1}(\eta_V)$ is the union of a curve and a finite number points.

There may be more than one irreducible curve in the fiber $\psi_X^{-1}(\eta_V)$. Proposition 3.9 can sometimes show the curve in $\psi_X^{-1}(\eta_V)$ is irreducible.

Proposition 4.1. *Suppose that A is a division k-algebra of index n. Define $f(n)$ to be the following function.*

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

Assume that $V \subset \mathbf{Hilb}^{\mathrm{tw}}_{f(n)t+s}(X/k)$ is an irreducible component and let $V_{sm} \subset V$ denote the locus of points smooth in V.

If V has a smooth k-rational point, i.e. $V_{sm}(k) \neq \emptyset$, then $\psi_X^{-1}(\eta_V)$ contains a unique irreducible and geometrically connected curve.

Proof. Because of the isomorphisms

$$\mathbf{Hilb}^{\mathrm{tw}}_{f(n)t+s}(X/k) \times_k k(\eta_V) \cong \mathbf{Hilb}^{\mathrm{tw}}_{f(n)t+s}(X_{k(\eta_V)}/k(\eta_V)),$$

the generic point $k(\eta_V)$ corresponds to a subscheme $\psi_X^{-1}(\eta_V) \subset X_{k(\eta_V)}$ with reduced Hilbert polynomial $\mathrm{rh}_V(t) = f(n)t + s$.

The Severi–Brauer variety $X_{k(\eta_V)}$ is associated to the central simple algebra $A_{k(\eta_V)}$ which, because of the assumption that $V_{sm}(k) \neq \emptyset$, has index n as well (apply Lemma A.1 to the Azumaya algebra $A \otimes_k \mathcal{O}_{V,x}$ where $x \in V_{sm}(k)$). Now the claim follows from Proposition 3.9. $\square$

Of particular interest is the following component of $\mathbf{Hilb}^{\mathrm{tw}}_{mt}(X/k)$ for any integer $m \geq 1$ such that n divides m.

Definition 4.2. Let $\mathrm{Ell}_m(X) \subset \mathbf{Hilb}^{\mathrm{tw}}_{mt}(X/k)$ denote the union of the irreducible components V of $\mathbf{Hilb}^{\mathrm{tw}}_{mt}(X/k)$ whose generic fiber $\psi_X^{-1}(\eta_V)$ is a smooth and geometrically connected curve of genus 1.

If either $\dim(X) = 2$ and $m = 3$, or if $\dim(X) \geq 3$ and $m \geq 4$ is an arbitrary integer divisible by n, then the scheme $\mathrm{Ell}_m(X)$ is nonempty. To see this, note that over an algebraic closure $\overline{k}$ one can always find a smooth genus 1 curve C; if D is any closed point on C, then the complete linear system associated to the divisor mD gives a closed immersion to $\mathbb{P}^{m-1}_{\overline{k}}$. If $m \leq \dim(X)$ then a linear embedding of $\mathbb{P}^{m-1}_{\overline{k}}$ in $\mathbb{P}^{d-1}_{\overline{k}}$ preserves the degree of C. Otherwise, a general projection from $\mathbb{P}^{m-1}_{\overline{k}}$ to $\mathbb{P}^{d-1}_{\overline{k}}$ is an embedding on C preserving the degree of C.

Let x be any closed point of $H = \mathbf{Hilb}^{\mathrm{tw}}_{mt}(X/k)$ defined by a linear system as described above. By base change we get a morphism

$$\psi_X|_{\mathscr{C}} : \mathscr{C} = \psi_X^{-1}(\mathrm{Spec}(\mathcal{O}_{H,x})) \to \mathrm{Spec}(\mathcal{O}_{H,x})$$

with special fiber $\mathscr{C}_{k(x)}$ geometrically isomorphic with the curve C. By [Sta19, Tag 01V9] there is an open $U' \subset \mathrm{Spec}(\mathcal{O}_{H,x})$ and an open $U \subset \mathscr{C}$ containing $\mathscr{C}_{k(x)} \subset U$ such that the restriction

$$\psi_X|_U : U \to U'$$

is smooth. The complement $\mathscr{C} \setminus U$ is closed in $\mathscr{C}$ and, since the map $\psi_X|_{\mathscr{C}}$ is proper, the image $\psi_X|_{\mathscr{C}}(\mathscr{C} \setminus U)$ is closed in $\mathrm{Spec}(\mathcal{O}_{H,x})$.

Now every nonempty closed subset of $\mathrm{Spec}(\mathcal{O}_{H,x})$ contains x and, by construction, the set $\psi_X|_{\mathscr{C}}(\mathscr{C} \setminus U)$ is closed and does not contain x. Thus $\psi_X|_{\mathscr{C}}(\mathscr{C} \setminus U) = \emptyset$ and necessarily $\mathscr{C} = U$ so that $\psi_X|_{\mathscr{C}}$ is smooth. Since $\psi_X|_{\mathscr{C}}$ is a smooth, proper, flat, and finitely presented morphism it follows from [Sta19, Tag 0E0N] that $\psi_X|_{\mathscr{C}}$ has geometrically connected fibers. The generic fiber $\psi_X^{-1}(\eta_V)$ for any irreducible component $V \subset H$ containing x is therefore a smooth and geometrically connected curve of genus 1 implying that $V \subset \mathrm{Ell}_m(X)$.

Proposition 4.3. *Suppose A is a central simple k-algebra of degree d and of index n. Then the following are true:*

(1) $\mathrm{Ell}_d(X)$ *is geometrically irreducible with* $\dim(\mathrm{Ell}_d(X)) = d^2$*;*

(2) if A has division and either A is cyclic or, if A contains a maximal subfield $F \subset A$ whose Galois closure E/k is a Galois extension of degree $2n$ with dihedral Galois group, then $\mathrm{Ell}_n(X)(k) \neq \emptyset$.

Proof. We first prove (2). In either case, let x be a point of X with $k(x)$ either a cyclic Galois extension E/k of k of degree n (in the first case) or a maximal subfield $k(x) \subset A$ with Galois closure E/k a dihedral Galois extension of degree $2n$ (in the second case). The field E splits X and $k(x) \otimes_k E \cong E^{\oplus n}$ either way. Let $H \subset \mathrm{Gal}(E/k)$ be a cyclic subgroup of order n. Pick an E-rational point p in x_E and let L be the line through p and gp for any generator g of H.

The union of the H-translates of L forms a $\mathrm{Gal}(E/k)$-orbit which descends to a scheme $V \subset X$ defined over k. Geometrically, the scheme $V_{\overline{k}}$ is an n-gon of lines through the points $x_{\overline{k}}$. Hence $\mathrm{rh}_V(t) = nt$. We claim the point defined by V in $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$ is contained in $\mathrm{Ell}_n(X)$.

Actually, as $V_{\overline{k}}$ is the scheme-theoretic union of lines we can use the exact sequence [Sta19, Tag 0C4J]

$$0 \to \mathcal{O}_{C\cup D} \to \mathcal{O}_C \oplus \mathcal{O}_D \to \mathcal{O}_{C\cap D} \to 0 \tag{7}$$

where $V_{\overline{k}} = C \cup D$, with C a chain of $n-1$ lines and D a line closing the n-gon, to compute that $h^1(V, \mathcal{O}_V) = 1$ and that $h^1(V_{\overline{k}}, \mathcal{O}_{V_{\overline{k}}}(1)) = 0$ by tensoring the exact sequence with $\mathcal{O}_{X_{\overline{k}}}(1)$. Since $V_{\overline{k}}$ has lci singularities, one can apply [Har10, Proposition 29.9] to find that $V_{\overline{k}}$ is smoothable.

More precisely, we find that $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$ is smooth at the k-rational point defined by $V \subset X$ and, over an algebraic closure, there is an integral curve passing through both the point corresponding to $V_{\overline{k}} \subset X_{\overline{k}}$ and the subset of $\mathrm{Ell}_n(X_{\overline{k}})$ parametrizing smooth and connected curves. In particular, the embedding $V \subset X$ defines a point of $\mathrm{Ell}_n(X)(k)$ completing the proof of (2).

Now we prove (1). If $d = 3$ then $\mathbf{Hilb}^{\mathrm{tw}}_{3t}(X/k)$ is a form of $\mathbb{P}^9$. However, we can be more precise. Since $d = 3$, the k-central simple k-algebra A associated to X is cyclic [KMRT98, Theorem 19.2]. If A is split, then so is $\mathbf{Hilb}^{\mathrm{tw}}_{3t}(X/k)$. Otherwise, A is a division k-algebra and, by part (2) above, there is a k-rational point $\mathbf{Hilb}^{\mathrm{tw}}_{3t}(X/k)(k) \neq \emptyset$. Hence $\mathrm{Ell}_3(X) = \mathbf{Hilb}^{\mathrm{tw}}_{3t}(X/k) \cong \mathbb{P}^9$ in this case too. (Alternatively, one can avoid the use of (2) by using Bertini's theorem).

So we can assume $d > 3$. Let $U' \subset \mathbf{Hilb}^{\mathrm{tw}}_{dt}(X/k) \times_k \overline{k}$ be the open set consisting of all smooth and connected curves of degree d and genus 1. Write U for the image of U' inside $\mathbf{Hilb}^{\mathrm{tw}}_{dt}(X/k)$. Since $d > 3$, it follows from [Ein86, Theorem 8] that U', and hence also $U \subset \mathrm{Ell}_d(X)$, is open and irreducible. We'll show that U is dense in $\mathrm{Ell}_d(X)$; this will prove the first claim since $U' = U \times_k \overline{k}$.

Let V be an irreducible component of $\mathrm{Ell}_d(X)$. Since the generic fiber $\psi_X^{-1}(\eta_V)$ is smooth and geometrically connected, there is an open subset $W \subset V$ such that for any point x in W, the curve $\psi_X^{-1}(x)$ is smooth (see [Gro67, Proposition 17.7.11]) and geometrically irreducible (by [Sta19, Tag 0559]) of degree d by assumption and, by Remark 3.6, of genus 1. In particular, we have $W \subset U$ showing that U is (topologically, but possibly not scheme-theoretically) dense in $\mathrm{Ell}_d(X)$.

The dimension of $\mathrm{Ell}_d(X)$ can be determined geometrically, i.e. over an algebraic closure, and this is done in [Ein86, Theorem 8]. Essentially, if $C \subset X_{\overline{k}}$ is smooth of degree d and genus 1 then one can compute

$$h^0(C, \mathcal{N}_{C/X_{\overline{k}}}) = d^2 \quad \text{and} \quad h^1(C, \mathcal{N}_{C/X_{\overline{k}}}) = 0$$

using the normal bundle sequence (and the Euler sequence for $X_{\overline{k}}$). This shows both that $\dim(\mathrm{Ell}_d(X)) \leq d^2$, from Corollary 2.9 (1), and that $\dim(\mathrm{Ell}_d(X)) \geq d^2$, from Corollary 2.9 (3); moreover this shows that $\mathrm{Ell}_d(X)$ is smooth along U. $\square$

Remark 4.4. The proof of (2) in Proposition 4.3 above is an extension of an argument due to Jason Starr, cf. [Sta17]. There Starr's goal is to use the fact that V defines a smooth k-rational point on $\mathbf{Hilb}^{\mathrm{tw}}_{nt}(X/k)$ to construct a smooth genus 1 curve on any Severi–Brauer variety X defined over a large (also called ample) field k (e.g. a p-special field or the fraction field of a Henselian DVR).

We can elaborate on Starr's result in the setting of Proposition 4.3, i.e. when A is a division k-algebra satisfying the assumptions of (2). Indeed, the scheme $\mathrm{Ell}_n(X)$ is projective so we can construct a smooth curve E with a k-rational point mapping to the k-point x associated to the n-gon V constructed in the proof of Proposition 4.3 (2) as follows.

Let y be any point of $\mathrm{Ell}_n(X)$ whose associated subscheme $C \subset X$ is a smooth geometrically connected curve of genus 1. Let $I = \{x, y\}$. Consider the blowup $\mathrm{Bl}_I(\mathrm{Ell}_n(X))$ with center the points $I \subset \mathrm{Ell}_n(X)$. Since $\mathrm{Ell}_n(X)$ is projective, there is some embedding of the blowup $\mathrm{Bl}_I(\mathrm{Ell}_n(X)) \subset \mathbb{P}^M$. A general linear section of the correct codimension intersects $\mathrm{Bl}_I(\mathrm{Ell}_n(X))$ in a curve (smooth near x) by Bertini's theorem [Jou83, Théorème 6.10 et Corollaire 6.11]. A general section of the same codimension intersects the exceptional divisor $\mathbb{P}^{n^2-1} \subset \mathrm{Bl}_I(\mathrm{Ell}_n(X))$ over x in a k-rational point and the exceptional divisor over y in some number of points. So we can choose a section $E' \subset \mathrm{Bl}_I(\mathrm{Ell}_n(X))$ doing all three things at once. The normalization E of E' is a curve with all the stated properties.

Over a large (also called ample) field k, any irreducible curve having a smooth k-rational point has infinitely many k-rational points. Thus the curve E has infinitely many k-rational points and the image along the composition of the normalization and blowdown

$$E \to E' \to \mathrm{Bl}_I(\mathrm{Ell}_n(X)) \to \mathrm{Ell}_n(X)$$

has nontrivial intersection with the open subset of $\mathrm{Ell}_n(X)$ consisting of smooth and geometrically connected genus 1 curves.

Example 4.5. If A is a cyclic division k-algebra of index n, there are lots of field extensions F/k where X_F contains a smooth geometrically connected curve of genus 1 and where the algebra A_F has index n. When $n = p^r$ is a power of a prime p, Remark 4.4 shows this holds for a minimal p-special field F/k contained in an algebraic closure $\overline{k}/k$. When the index n is arbitrary one can instead use the field $k((t))$, which is the fraction field of a Henselian DVR, and apply Remark 4.4. The index remains n here since $A_{k((t))}$ specializes to A (Lemma A.1).

One can also construct "generic" examples for an arbitrary division algebra A of index n as follows. If $n = p$ is prime, then one can first replace the base field k by an extension F/k with A_F a cyclic division algebra of index p if necessary. Next, one again extends the base field but now to the function field $L = F(\mathrm{Ell}_p(X_F))$ of the scheme $\mathrm{Ell}_p(X_F)$. Since $\mathrm{Ell}_p(X_F)$ has a smooth F-rational point by Proposition 4.3, the algebra A_L is nonsplit (hence of index p) by [GS17, Lemma 5.4.7]. The generic fiber $\psi_{X_F}^{-1}(\eta_{\mathrm{Ell}_p(X_F)})$ is then a smooth and geometrically connected genus 1 curve on X_L.

If n is not prime, one can use [RTY08] to get a field extension F/k with A_F cyclic of index n and with the restriction $\mathrm{Br}(k) \to \mathrm{Br}(F)$ an injection. Setting $L = F(\mathrm{Ell}_n(X_F))$ then, as above, X_L contains a smooth and geometrically connected curve of genus 1. In this situation [GS17, Lemma 5.4.7] shows that the restriction $\mathrm{Br}(F) \to \mathrm{Br}(L)$ is an injection and Lemma A.1 below shows that A_L remains index n (actually, both statements can be obtained from Lemma A.1). Hence also the extension of A to $E = k(\mathrm{Ell}_n(X))$ has index n and X_E contains a smooth and geometrically connected curve of genus 1.

Example 4.6. Let $n \geq 3$ be an integer and fix a divisor $m \geq 1$ of n. Set $G = \mathrm{SL}_n/\mu_m$ to be the quotient of the special linear group by the sub-group scheme of mth roots of unity. Fix a faithful representation $G \to \mathrm{GL}_N$ for some $N \gg 0$ and let $\pi : \mathrm{GL}_N \to \mathrm{GL}_N/G$ be the quotient. If $P \subset G$ is a parabolic subgroup such that $P\backslash G \cong \mathbb{P}^{n-1}$, then π is equivariant for the right-action of P and the quotient by this action yields a Severi–Brauer scheme $\pi_0 : P\backslash\mathrm{GL}_N \to \mathrm{GL}_N/G$. One can therefore consider the relative GL_N/G-scheme $\mathbf{Hilb}^{tw}_{nt}(\pi_0)$ and, if η is the generic point of the (smooth and geometrically irreducible) scheme GL_N/G, we can define the relative GL_N/G-scheme $\mathrm{Ell}_n(\pi_0)$ as the scheme theoretic closure of $\mathrm{Ell}_n(\pi_0 \times_{\mathrm{GL}_N/G} \eta)$ inside $\mathbf{Hilb}^{tw}_{nt}(\pi_0)$.

The scheme $\mathrm{Ell}_n(\pi_0)$ is proper and surjective over GL_N/G and, for any field extension F/k and for any F-point $x \in (\mathrm{GL}_N/G)(F)$, the fiber $\mathrm{Ell}_n(\pi_0) \times_{\mathrm{GL}_N/G} x$ contains $\mathrm{Ell}_n(\pi_0 \times_{\mathrm{GL}_N/G} x)$ as a closed subscheme. By [Sta19, Tag 0559], there is then an open subscheme $W \subset \mathrm{GL}_N/G$ such that for any $x \in W(F)$ there is an equality

$$\mathrm{Ell}_n(\pi_0) \times_{\mathrm{GL}_N/G} x \cong \mathrm{Ell}_n(\pi_0 \times_{\mathrm{GL}_N/G} x).$$

If the base field k is infinite, then the relative Severi–Brauer scheme π_0 is versal (cf. [GMS03, Ch. 1 §5]) in the sense that for any nonempty open subscheme $U \subset \mathrm{GL}_N/G$, for any field extension F/k, and for any Severi–Brauer variety X associated to an F-central simple F-algebra A with $\deg(A) = n$ and $\exp(A)$ dividing m, there exists an F-point $x \in$

$U(F)$ so that $X \cong \pi_0^{-1}(U) \times_U x$. The scheme $\mathrm{Ell}_n(\pi_0) \times_{\mathrm{GL}_N/G} W$ and its universal family, considered over W, is similarly versal for geometrically elliptic normal curves on Severi–Brauer varieties.

Moreover, using Example 4.5, there exists a *generic geometrically elliptic normal curve* $C^{gen}_{n,m}$ on the base extension $(X^{gen}_{n,m})_E$ of the generic Severi–Brauer variety $X^{gen}_{n,m} = \pi_0^{-1}(\eta)$, where E is the function field of the scheme $\mathrm{Ell}_n(\pi_0 \times_{\mathrm{GL}_N/G} \eta)$. Fix any field F/k, fix a point $x \in W(F)$ corresponding to a Severi–Brauer variety X, and fix a geometrically elliptic normal curve $C \subset X$. The point s_x in $S = \mathrm{Ell}_n(\pi_0 \times_{\mathrm{GL}_N/G} x)$ associated to the subscheme $C \subset X$ is geometrically regular. Hence there exists a sequence of DVRs $(R_0, \mathfrak{m}_0), ..., (R_{j(s_x)}, \mathfrak{m}_{j(s_x)})$ satisfying the following conditions:

(1) $\mathrm{Frac}(R_0) = F(\mathrm{Ell}_n(\pi_0)|_W) := E'$,
(2) $R_i/\mathfrak{m}_i \cong \mathrm{Frac}(R_{i+1})$
(3) $R_{j(s_x)}/\mathfrak{m}_{j(s_x)} \cong F(s_x)$.

There are also smooth $\mathrm{Spec}(R_i)$-schemes, gotten by base change of the universal family, which at one end gives $C^{gen}_{n,m} \times_E E'$ and the other C. In this way the generic geometrically elliptic normal curve specializes to any other geometrically elliptic normal curve in any Severi–Brauer variety defined over any field extension of k.

Recall that the period $\mathrm{per}(C)$ of a smooth, proper, and geometrically integral curve C/k is the smallest integer $m \geq 1$ so that $\mathbf{Pic}^m_{C/k}(k) \neq \emptyset$. Equivalently, the period of C/k is the order of the element $[\mathbf{Pic}^1_{C/k}]$ inside the first Galois cohomology group $\mathrm{H}^1(k, \mathbf{Pic}^0_{C/k})$.

Recall also that the index $\mathrm{ind}(C)$ of C is the unique positive integer generating the image of the degree map $\deg : \mathrm{CH}_0(C) \to \mathbb{Z}$. We have that $\mathrm{per}(C)$ divides $\mathrm{ind}(C)$ and if the genus of C satisfies $g(C) = 1$, then $\mathrm{ind}(C)$ divides $\mathrm{per}(C)^2$, see [Lic69, Theorem 8]. In the following theorem we keep the notation of Example 4.6 (in particular, the base field k is assumed to be infinite).

Theorem 4.7. *Let $n \geq 3$ be an integer, and let $m > 1$ be a divisor of n such that n and m have the same prime factors (i.e. $m \mid n \mid m^\infty$). Assume, additionally, that n is not divisible by the characteristic of k.*

Then the generic geometrically elliptic normal curve $C^{gen}_{n,m}$ above has index $\mathrm{ind}(C^{gen}_{n,m}) = nm$ *and* $\mathrm{per}(C^{gen}_{n,m}) = n$.

Remark 4.8. Let $A^{gen}_{n,m}$ be the central simple $k(\eta)$-algebra associated to the generic Severi–Brauer variety $X^{gen}_{n,m}$. If $n = st$ is a factorization by integers s and t such that $\gcd(t, m) = 1$ and s and m share the same prime factors, then

$$\deg(A_{n,m}^{gen}) = n, \quad \mathrm{ind}(A_{n,m}^{gen}) = s, \quad \text{and} \quad \exp(A_{n,m}^{gen}) = m.$$

So the assumptions on n and m in Theorem 4.7 describe, equivalently, exactly those cases where $A_{n,m}^{gen}$ is a division algebra.

Proof. We first deal with the case when $n = m$. Since $C_{n,n}^{gen}$ embeds as a geometrically elliptic normal curve in a Severi–Brauer variety of dimension $n-1$, we find $\mathrm{per}(C_{n,n}^{gen})$ divides n. To prove $\mathrm{per}(C_{n,n}^{gen}) = n$, it therefore suffices to show $\mathrm{ind}(C_{n,n}^{gen}) = n^2$.

Because of our assumption that the characteristic of the base field k does not divide n, we can find a field extension F/k and a smooth, proper, and geometrically integral F-curve C of genus $g(C) = 1$ with $\mathrm{ind}(C) = n^2$ and $\mathrm{per}(C) = n$. (By the remarks at the end of §4 in [LT58], one can take $F = \bar{k}((t_1))((t_2))$ for an algebraic closure $\bar{k}$ of k). After base extension from k to F, it follows that $(C_{n,n}^{gen})_{E'}$ specializes (along a sequence of DVRs) to C as above; here E' is the function field of $\mathrm{Ell}_n(\pi_0 \times_k F)$ as before. Hence, by [Ful98, Proposition 20.3 (a)], the index $\mathrm{ind}(C_{n,n}^{gen})$ is divisible by n^2 which implies that it actually is n^2.

When $n \neq m$, we can similarly argue by specialization. In this case, we still have $\mathrm{per}(C_{n,m}^{gen})$ divides n since $C_{n,m}^{gen}$ embeds in a Severi–Brauer variety of dimension $n-1$ as a geometrically elliptic normal curve. Since n is indivisible by the characteristic of k, we can construct (see Lemma 4.9 below) a smooth, proper, and geometrically integral curve C over a field extension F/k with $\mathrm{per}(C) = \mathrm{ind}(C) = n$. This curve C embeds as a geometrically elliptic normal curve on the trivial Severi–Brauer variety $\mathbb{P}_F^{n-1}$, which is associated to a central simple F-algebra of degree n and exponent dividing m trivially. Hence we can specialize $(C_{n,m}^{gen})_{E'}$ to C along a sequence of DVRs; here $E' = F(\mathrm{Ell}_n(\pi_0 \times_k F))$. As each relative curve that appears over a DVR in this process is projective, smooth, and has geometrically integral fibers, we can consider their associated Picard schemes [Kle05, Theorem 4.8]. In this way we can also specialize from $\mathbf{Pic}^d_{(C_{n,m}^{gen})_{E'}/E'} \cong \mathbf{Pic}^d_{C_{n,m}^{gen}/E} \times_E E'$ to $\mathbf{Pic}^d_{C/F}$, for each integer d dividing n, along a sequence of DVRs. Since the period can only decrease when extending the base field, we can apply [Ful98, Proposition 20.3 (a)] to show that $\mathrm{per}(C_{n,m}^{gen}) = n$ as claimed.

To compute the index of $C_{n,m}^{gen}$, we also use a specialization argument. Let A be the central simple E-algebra corresponding to $(X_{n,n}^{gen})_E$ and let $X = \mathbf{SB}(A^{\otimes m})$. Since $(C_{n,n}^{gen})_{E(X)}$ sits on the Severi–Brauer variety $(X_{n,n}^{gen})_{E(X)}$, which is associated to the division algebra $A_{E(X)}$ of index n and exponent m by [SVdB92, Theorem 2.1], we can specialize $(C_{n,m}^{gen})_{E'}$ to this curve along a sequence of DVRs; $E' = E(X)(\mathrm{Ell}_n(\pi_0 \times_k E(X)))$.

We show in Lemma 4.10 below that the curve $(C^{gen}_{n,n})_{E(X)}$ has index nm. Thus, using [Ful98, Proposition 20.3 (a)] again, we get $\mathrm{ind}(C^{gen}_{n,m}) \geq nm$. However, it's possible to see that we must also have $\mathrm{ind}(C^{gen}_{n,m}) \leq nm$ as we now explain.

Indeed, if B is the central simple $k(\eta)$-algebra associated to $X^{gen}_{n,m}$ then B has index n and exponent m. If $E = k(\mathrm{Ell}_n(\pi_0))$ is the given function field, then $(X^{gen}_{n,m})_E$ is associated to the algebra B_E which still has index $\mathrm{ind}(B_E) = n$ and exponent $\exp(B_E) = m$ as the restriction $\mathrm{Br}(k(\eta)) \to \mathrm{Br}(E)$ is an injection (see Example 4.5). If H is a divisor of $(X^{gen}_{n,m})_E$ of degree $\exp(B_E) = m$ then

$$[C^{gen}_{n,m} \cap H] = [C^{gen}_{n,m}][H] = m[p]$$

holds in $\mathrm{CH}_0((X^{gen}_{n,m})_E)$ for some point p of degree $\mathrm{ind}(B_E) = n$. Now the left hand side of this equation has degree some multiple of the index of $C^{gen}_{n,m}$ whereas the right hand side has degree nm. □

We needed two lemmas for the above proof. The first of these lemmas constructs curves of equal period and index over an extension of k. The proof below is adapted from [use].

Lemma 4.9. *Let $n \geq 1$ be an integer not divisible by the characteristic of k. Let $\bar{k}$ be a fixed algebraic closure of k. Write $F = \bar{k}((t))$ for the field of formal Laurent series in t over $\bar{k}$. Then there exists a smooth and proper genus one curve C/F with $\mathrm{per}(C) = \mathrm{ind}(C) = n$.*

Proof. Let E/k be any elliptic curve. We claim that there exists an element $x \in \mathrm{H}^1(F, E_F)$ having exact order n. Using the correspondence between this Galois cohomology group and the Weil–Châtelet group for E_F, the element x corresponds to an E_F-torsor C/F having period n. By [Lic68, Theorem 1], the curve C also has index n.

The Kummer sequence associated to the multiplication-by-n map on E_F yields the exact sequence

$$0 \to E_F(F)/nE_F(F) \to \mathrm{H}^1(F, E_F[n]) \to \mathrm{H}^1(F, E_F)[n] \to 0 \tag{8}$$

where $E_F[n]$ is the subgroup scheme of n-torsion points of E_F. Since n is not divisible by the characteristic of k, and since E is defined over k, there exists an isomorphism of group schemes $E_F[n] \cong (\mathbb{Z}/n\mathbb{Z})^{\oplus 2}$. Since F admits a cyclic Galois extension of degree n (i.e. $F(t^{1/n})$), there exists an element $z \in \mathrm{H}^1(F, E_F[n])$ of exact order n.

We claim that the group $E_F(F)/nE_F(F) = 0$ so that, by (8), there exists an element x of order n as desired (the image of z, for example). Let $R = \bar{k}[[t]]$. The restriction $E_R(R) \to E_F(F)$ is an isomorphism due to the valuative criterion for properness, so it suffices to show that

$E_R(R)$ is n-divisible. Since E is finitely presented over k and we have that $R = \varprojlim_m R/(t^m)$, there is an isomorphism

$$\varinjlim_m E_{R/(t^m)}(R/(t^m)) \cong \varinjlim_m E_R(R/(t^m)) \cong E_R(R).$$

We'll show that $E_{R/(t^m)}(R/(t^m))$ is n-divisible by induction on m. When $m = 1$, the group $E_{\bar{k}}(\bar{k})$ is divisible as E is an elliptic curve. Now assume $E_{R/(t^m)}(R/(t^m))$ is n-divisible for some $m \geq 1$. From the restriction we get an exact sequence

$$0 \to V \to E_R(R/(t^{m+1})) \to E_R(R/(t^m)) \to 0$$

with surjectivity on the right by formal smoothness. Here the kernel V is a $\bar{k}$-vector space which is n-divisible since the characteristic of k doesn't divide n. It follows that $E_R(R/(t^{m+1})) = E_{R/(t^{m+1})}(R/(t^{m+1}))$ is n-divisible as well. $\square$

The second lemma provides an index reduction formula for the generic curve $C^{gen}_{n,n}$.

Lemma 4.10. *Let $n \geq 3$ be an integer not divisible by the characteristic of the base field k and fix a divisor $m \geq 1$ of n sharing the same prime factors as n if $m > 1$. Let A be the central simple E-algebra associated to the Severi–Brauer variety $(X^{gen}_{n,n})_E$. Let $X = \mathbf{SB}(A^{\otimes m})$.*

Then the generic geometrically elliptic normal curve $C^{gen}_{n,n} \subset (X^{gen}_{n,n})_E$ satisfies $\operatorname{ind}\big((C^{gen}_{n,n})_{E(X)}\big) = nm$. Moreover, if n/m is squarefree, then the period of $(C^{gen}_{n,n})_{E(X)}$ is $\operatorname{per}\big((C^{gen}_{n,n})_{E(X)}\big) = n$.

Proof. Let $C = C^{gen}_{n,n}$ for the proof. Now there exists an exact sequence

$$0 \to \operatorname{Pic}(C \times X) \to \mathbf{Pic}_{C\times X/E}(E) \xrightarrow{\delta} \operatorname{Br}(E) \tag{9}$$

which can be obtained in multiple ways, see for example [CK12, Proof of Theorem 2.1] or [Kle05, Remark 2.11]. Important for us are the facts that there is an equality

$$\mathbf{Pic}_{C\times X/E}(E) = \operatorname{Pic}((C \times X)_{E^s})^{\operatorname{Gal}(E^s/E)},$$

where E^s is a separable closure of E, and that there is a geometric realization of the rightmost map of (9), see e.g. [Lie17, Theorem 3.4].

Using the above equality, we can compute $\mathbf{Pic}_{C\times X/E}(E)$ explicitly. There is an exact sequence of $\operatorname{Gal}(E^s/E)$-modules

$$0 \to \operatorname{Pic}(C_{E^s}) \times \operatorname{Pic}(X_{E^s}) \to \operatorname{Pic}((C \times X)_{E^s}) \to H \to 0$$

where $H = \operatorname{Hom}\big(\big(\mathbf{Pic}^0_{X_{E^s}/E^s}\big)^\vee, \mathbf{Pic}^0_{C_{E^s}/E^s}\big)$ and the leftmost nonzero map is the pullback along the two projections, see [CTS21, §5.7.1].

Note that, as X is a Severi–Brauer variety, we have $H = 0$ so that

$$\mathbf{Pic}_{C\times X/E}(E) \cong (\mathrm{Pic}(C_{E^s}) \times \mathrm{Pic}(X_{E^s}))^{\mathrm{Gal}(E^s/E)} \cong \mathbf{Pic}_{C/E}(E) \times \mathbb{Z}$$

where a generator in the second component is given by the class of the line bundle $\mathcal{O}(1)$ on $X_{E^s} \cong \mathbb{P}^{n-1}_{E^s}$. Note that $\mathcal{O}(1)$ maps to $[A^{\otimes m}]$ in the Brauer group $\mathrm{Br}(E)$ under the map δ.

Suppose an E-rational point in $\mathbf{Pic}_{C\times X/E}(E)$ is given by the pair $x = (\mathcal{L}, \mathcal{O}(-\ell))$. Then x comes from a line bundle on $C \times X$ only if its image in $\mathrm{Br}(E)$ is trivial, i.e. if there is an equality

$$0 = \delta(\mathcal{L}) - [A^{\otimes m\ell}].$$

Since C has an E-rational divisor of degree n^2, whose image in $\mathrm{Br}(E)$ is trivial, we can translate such an x to a pair where the first component has degree $0 < d = \deg(\mathcal{L}) \leq n^2$. We can even assume $d > 1$ since $C \cong \mathbf{Pic}^1_{C/E}$ has no E-rational points.

Since C has genus $g(C) = 1$, any line bundle representing the point $\mathcal{L}$ on $\mathbf{Pic}^d_{C/E}(E)$ is globally generated and thus defines a morphism

$$\varphi : C \to P$$

where P is a Severi–Brauer variety with class $[P] = \delta(\mathcal{L}) = [A^{\otimes m\ell}]$ in $\mathrm{Br}(E)$ by [Lie17, Theorem 3.4]. If D is a Weil divisor on P of degree $e = \exp(A^{\otimes m\ell})$, then the zero-cycle $[C \cap D] \in \mathrm{CH}_0(C)$ has degree de considered as a Weil divisor of C. Since $\mathrm{ind}(C) = n^2$, we must have n^2 divides de. Since $\exp(A^{\otimes m\ell})$ divides n/m, we get

$$n^2 \mid de \mid d\left(\frac{n}{m}\right).$$

Hence nm divides d.

Now there is a commutative box (all faces commute)

$$\begin{tikzcd}
 & \mathrm{Pic}(C\times X) \arrow[rr] \arrow[dl, two heads] \arrow[dd] & & \mathrm{Pic}((C\times X)_{E^s}) \arrow[dl, two heads] \arrow[dd] \\
\mathrm{Pic}(C_{E(X)}) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Pic}((C_{E^s})_{E^s(X_{E^s})}) & \\
 & \mathrm{CH}^0(X) \arrow[rr, equal] \arrow[dl, equal] & & \mathrm{CH}^0(X_{E^s}) \arrow[dl, equal] \\
\mathbb{Z} \arrow[rr, equal] & & \mathbb{Z} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where all vertical arrows are pushforward morphisms, all other arrows are pullbacks, and we've identified

$$\mathrm{CH}_0(\mathrm{Spec}(E(X))) = \mathbb{Z} = \mathrm{CH}_0(\mathrm{Spec}(E^s(X_{E^s}))),$$

see [Ful98, Proposition 1.7]. By localization [EKM08, Corollary 57.11], all slanted arrows are surjective and the bottom square is trivially all isomorphisms.

If $\mathcal{L}_0$ is a line bundle on $C_{E(X)}$ we can therefore lift it to a line bundle on $C \times X$ which, over the separable closure E^s/E is of the form $x = (\mathcal{L}, \mathcal{O}(-\ell))$ for a line bundle $\mathcal{L}$ on C_{E^s} with $\deg(\mathcal{L})$ a multiple of nm and for some $\ell \in \mathbb{Z}$. By pushing forward to X_{E^s} and restricting to $E^s(X_{E^s})$, we see that $\deg(\mathcal{L}_0) = \deg(\mathcal{L})$ is a multiple of nm. Conversely, taking m-times the point of $\mathbf{Pic}^n_{C/E}$ corresponding to the embedding $C \subset X^{gen}_{n,n}$ defines a degree nm line bundle on $C_{E(X)}$. Hence $\mathrm{ind}(C_{E(X)}) = nm$.

Finally, assume that n/m is squarefree. If the period of $C_{E(X)}$ is d, then d divides n since the period of C was n. If $d \neq n$, then there is a prime p so that $v_p(d) \leq v_p(n) - 1$ where v_p is the p-adic valuation. Now we have d divides nm which divides d^2 by the period/index relations. However, if n/m is squarefree then $v_p(m) \geq v_p(n) - 1$ so that

$$v_p(nm) = v_p(n) + v_p(m) \geq 2v_p(n) - 1$$

while $v_p(d^2) = 2v_p(d) \leq 2v_p(n) - 2$. Hence $d = n$. □

Appendix A. On Azumaya algebras

Lemma A.1. *Let R be a Noetherian regular local ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$, and fraction field F. Suppose that A is an Azumaya R-algebra. Then there is an inequality* $\mathrm{ind}(A_k) \leq \mathrm{ind}(A_F)$.

Proof. We consider the R-schemes $X_m = \mathbf{SB}_m(A)$ which are étale forms of the Grassmannian R-schemes $\mathbf{Gr}_R(m, n)$, where n is the square root of the rank of A, and for varying m. The F and k fibers of the structure map over R are canonically

$$\mathbf{SB}_m(A_F) \cong \mathbf{SB}_m(A) \times_R F \quad \text{and} \quad \mathbf{SB}_m(A_k) \cong \mathbf{SB}_m(A) \times_R k,$$

which have an F-rational point, or a k-rational point respectively, if and only if the index $\mathrm{ind}(A_F)$, or $\mathrm{ind}(A_k)$ respectively, divides m [Bla91, Proposition 3]. We'll show that the assumption R is regular guarantees that $\mathbf{SB}_m(A_k)(k) \neq \emptyset$ whenever $\mathbf{SB}_m(A_F)(F) \neq \emptyset$.

For this, we first note that R admits a sequence of discrete valuation rings $R_0, ..., R_t$ with maximal ideals $\mathfrak{m}_0, ..., \mathfrak{m}_t$ for some $t \geq 0$ with the following properties:

(1) $\mathrm{Frac}(R_0) = F$,
(2) $R_i/\mathfrak{m}_i \cong \mathrm{Frac}(R_{i+1})$
(3) $R_t/\mathfrak{m}_t \cong k$.

One can take a regular sequence $(a_0, ..., a_{t-1})$ of generators for $\mathfrak{m}$ and define $R_i = (R/(a_0, ..., a_{i-1}))_{(a_i)}$ (cf. [Sta19, Tag 00NQ, Tag 0AFS]). Now the valuative criterion for properness [Har66, Theorem 4.7] shows

$$(X_m)_{R_i}(\mathrm{Frac}(R_i)) \neq \emptyset \implies (X_m)_{R_{i+1}}(\mathrm{Frac}(R_{i+1})) \neq \emptyset.$$

One can conclude by induction. □

Example A.2. The assumption that R is regular cannot be dropped from the statement of Lemma A.1. Here's an example from [Ma22, §4]. Fix a field k. Let X/k be any Severi–Brauer variety having $X(k) = \emptyset$. Let $x \in X$ be a closed point. Consider the pushout $\tilde{X}$ in the cocartesian diagram below.

$$\begin{tikzcd} x \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ X \arrow[r] & \tilde{X} \end{tikzcd}$$

Let $\tilde{x} \in \tilde{X}$ denote the canonical (singular) k-rational point of $\tilde{X}$ and $\mathcal{O}_{\tilde{X},\tilde{x}}$ the local ring. If A is the central simple algebra associated to X, then the Azumaya algebra $A \otimes_k \mathcal{O}_{\tilde{X},\tilde{x}}$ is split over the generic point and nontrivial over the closed point by construction.

References

[Bla91] Altha Blanchet, *Function fields of generalized Brauer-Severi varieties*, Communications in Algebra **19** (1991), no. 1, 97–118.

[CK12] Mirela Ciperiani and Daniel Krashen, *Relative Brauer groups of genus 1 curves*, Israel J. Math. **192** (2012), no. 2, 921–949. MR 3009747

[CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov, *The Brauer-Grothendieck group*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 71, Springer, Cham, [2021] ©2021. MR 4304038

[Ein86] Lawrence Ein, *Hilbert scheme of smooth space curves*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 4, 469–478. MR 875083

[EKM08] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008. MR 2427530

[Ful98] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323

[GMS03] Skip Garibaldi, Alexander Merkurjev, and Jean-Pierre Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28, American Mathematical Society, Providence, RI, 2003. MR 1999383

[Gro67] Alexander Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 238860

[GS17] Philippe Gille and Tamás Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 165, Cambridge University Press, Cambridge, 2017, Second edition of [MR2266528]. MR 3727161

[Har66] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 5–48. MR 213368

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har81] Joe Harris, *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 1, 35–68. MR 616900

[Har10] Robin Hartshorne, *Deformation theory*, Graduate Texts in Mathematics, vol. 257, Springer, New York, 2010. MR 2583634

[Jou83] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983. MR 725671

[Kle05] Steven L. Kleiman, *The Picard scheme*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 235–321. MR 2223410

[KM19] Nikita Karpenko and Eoin Mackall, *On the K-theory coniveau epimorphism for products of Severi-Brauer varieties*, Ann. K-Theory **4** (2019), no. 2, 317–344, With appendices by Mackall. MR 3990787

[KMRT98] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits. MR 1632779

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180

[Lic68] Stephen Lichtenbaum, *The period-index problem for elliptic curves*, Amer. J. Math. **90** (1968), 1209–1223. MR 237506

[Lic69] ———, *Duality theorems for curves over p-adic fields*, Invent. Math. **7** (1969), 120–136. MR 242831

[Lie17] Christian Liedtke, *Morphisms to Brauer-Severi varieties, with applications to del Pezzo surfaces*, Geometry over nonclosed fields, Simons Symp., Springer, Cham, 2017, pp. 157–196. MR 3644253

[LT58] Serge Lang and John Tate, *Principal homogeneous spaces over abelian varieties*, Amer. J. Math. **80** (1958), 659–684. MR 106226

[Ma22] Qixiao Ma, *Addition of Brauer classes via Picard schemes*, J. Algebra **589** (2022), 105–113. MR 4321613

[Mac21] Eoin Mackall, *Algebraic connective K-theory of a Severi-Brauer variety with prescribed reduced behavior*, Preprint (2021), 1–14, Available on author's webpage: https://www.eoinmackall.com/s/revconnectivek2.pdf.

[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966, With a section by G. M. Bergman. MR 0209285

[Nit05] Nitin Nitsure, *Construction of Hilbert and Quot schemes*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 105–137. MR 2223407

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, 85–147. Lecture Notes in Math., Vol. 341. MR 0338129

[RS96] Louis H. Rowen and David J. Saltman, *Semidirect product division algebras*, Israel J. Math. **96** (1996), no. part B, 527–552. MR 1433706

[RTY08] U. Reman, S. V. Tikhonov, and V. I. Yanchevskiĭ, *Symbol algebras and the cyclicity of algebras after a scalar extension*, Fundam. Prikl. Mat. **14** (2008), no. 6, 193–209. MR 2533621

[Sta17] Jason Starr, *For each n: show there is a genus 1 curve over some field k with no points of degree less than n, (simple argument / best reference)?*, MathOverflow, 2017, https://mathoverflow.net/q/286458 (version: 2017-11-20).

[Sta19] The Stacks project authors, *The stacks project*, https://stacks.math.columbia.edu, 2019.

[SVdB92] A. Schofield and M. Van den Bergh, *The index of a Brauer class on a Brauer-Severi variety*, Trans. Amer. Math. Soc. **333** (1992), no. 2, 729–739. MR 1061778

[use] user117273, *For each n: show there is a genus 1 curve over some field k with no points of degree less than n, (simple argument / best reference)?*, MathOverflow, https://mathoverflow.net/q/286395 (version: 2017-11-18).

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