A DEGREE-BOUND ON THE GENUS OF A PROJECTIVE CURVE

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ABSTRACT. We give a short proof of the following statement: if C is a geometrically integral and projective curve with degree d and arithmetic genus g, defined over an arbitrary field k, then one has $g \leq d^2 - 2d + 1$. As a corollary, we show vanishing of the top cohomology of some twists of the structure sheaf of a geometrically integral projective variety.

Notation and Conventions. Throughout this note we write k for a fixed but arbitrary base field. A variety defined over k, or a k-variety, or simply a variety if the base field k is clear from context, is a separated scheme of finite type over k. A curve is a proper variety of pure dimension one.

1. Introduction

The observation that curves of a fixed degree have genus bounded by an expression in the degree of the curve is classical, dating back to an 1889 theorem of Castelnuovo. An account of this result, that holds for smooth and geometrically irreducible curves, was given by Harris in [Har81]; see also [Har77, Chapter 4, Theorem 6.4] for a particular case of this theorem.

In [GLP83], Gruson, Lazarsfeld, and Peskine give a bound similar in spirit to Castelnuovo's for geometrically integral projective curves. The most general form of their result states that for any such curve $C \subset \mathbb{P}^n$ of degree d and arithmetic genus g, one has $g \leq d^2 - 2d + 1$ but, this bound can be sharpened when C is assumed to be nondegenerate.

In this note we give a short proof of the bound $g \leq d^2 - 2d + 1$ on the arithmetic genus g of an arbitrary geometrically integral and curve $C \subset \mathbb{P}^n$ of degree d. Our proof is cohomological in nature and proceeds by establishing an effective bound $\epsilon = \epsilon(d) \geq 0$ for the vanishing of $\mathrm{H}^1(C, \mathcal{O}_C(\epsilon))$; this is the content of our main theorem, Theorem 3.3. By taking a suitable hyperplane section, we're able to relate ϵ to a bound $\delta = \delta(d)$ on the order of twisting that's needed to generate the global sections of a finite projective subscheme of $\mathbb{P}^n$ of degree d. We then utilize a lemma of Poonen [Poo04] that determines this latter bound explicitly.

Added: A deeper analysis of Hartshorne [Har94] classifies curves of maximal possible genus inside $\mathbb{P}^3$ using the same, or very similar, techniques.

As a consequence of the result given here on curves, we deduce a vanishing statement for the top cohomology of twists of the structure sheaf of a geometrically integral projective variety. For smooth varieties, this can be deduced from the main theorems of [Har81] by Serre duality.

The contents of this note are as follows. Section 2 is for background; here we recall some standard definitions and one result on the Hilbert polynomial of geometrically irreducible curves that's needed for the proof of our bound. Section three then is devoted to our main result and Section four is devoted to its corollary.

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2. Preliminaries

In this section we recall some results on the Hilbert polynomial of a projective variety $X \subset \mathbb{P}^n$.

Definition 2.1. Let $X \subset \mathbb{P}^n$ be a projective variety. Then there is a uniquely determined numerical polynomial $p_X(t)$ that agrees with the Euler characteristic

$$p_X(t) = \chi(X, \mathcal{O}_X(t))$$

for all $t \geq 0$. The polynomial $p_X(t)$ is called the *Hilbert polynomial of X with respect to the given embedding*. We may, at times, informally refer to the Euler characteristic itself as the Hilbert polynomial when no confusion should occur.

Definition 2.2. Let $X \subset \mathbb{P}^n$ be a projective variety of pure dimension r. The *degree* $\deg(X)$ of X under this embedding is defined as $r!$ times the leading coefficient of the Hilbert polynomial $p_X(t)$.

In the case of a geometrically integral and projective curve, the Hilbert polynomial is determined by its degree and the arithmetic genus of the curve.

Lemma 2.3. *Suppose that $C \subset \mathbb{P}^n$ is a geometrically integral projective curve. Then the Hilbert polynomial of C is the linear function*

$$p_C(t) = dt + 1 - g$$

where g is the arithmetic genus of C and $d = \deg(C)$ is the degree of C. □

3. The degree-bound

In this section we prove our main theorem, Theorem 3.3 below, that gives a bound on the arithmetic genus of a geometrically integral and projective curve C in terms of the degree of C. We start with two lemmas needed in the proof.

Lemma 3.1 ([Poo04, Lemma 2.1])**.** *Let $Y \subset \mathbb{P}^n$ be a subvariety that is finite over the base field k. Then the canonical morphism*

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(\delta)) \to \mathrm{H}^0(Y, \mathcal{O}(\delta))$$

is surjective for all $\delta \geq \dim \mathrm{H}^0(Y, \mathcal{O}_Y) - 1$. □

Lemma 3.2. *Assume that k is algebraically closed. Let $C \subset \mathbb{P}^n$ be a geometrically irreducible curve defined over k and of degree $d = \deg(C)$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ with the following properties:*

1. *$C \cap H$ contains no associated point of C,*
2. *$\dim \mathrm{H}^0(C \cap H, \mathcal{O}_{C \cap H}) = d$.*

Proof. Since C is locally noetherian, the set of embedded points of C is finite, [Sta19, Tag 05AF]. Since for any finite set of closed points $S \subset C$, the collection of hyperplanes in $\mathbb{P}^n$ that intersect S forms a closed subvariety of the Grassmannian $\mathrm{Gr}(n, n+1)$ of n dimensional linear spaces in a k-vectorspace of dimension $n+1$, there is an open subvariety $U \subset \mathrm{Gr}(n, n+1)$ corresponding to those hyperplanes that miss all associated points of C. By Bertini's theorem [Jou83, Théorèm 6.10 et Corollaire 6.11] the collection of hyperplanes in $\mathbb{P}^n$ whose intersection with C is a finite subscheme over k also forms an open subvariety W of $\mathrm{Gr}(n, n+1)$. As k is infinite, there exists a rational point in $U \cap W$ corresponding to a hyperplane $H \subset \mathbb{P}^n$ satisfying property (1).

To see that H satisfies property (2), we note there exists an exact sequence of sheaves on C

$$0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathcal{O}_{C \cap H} \to 0 \qquad \text{(S1)}$$

since H avoids the associated points of C. Twisting by $\mathcal{O}_C(m)$ for large $m \geq 0$ one finds that

$$\dim \mathrm{H}^0(C \cap H, \mathcal{O}_{C\cap H}) = \dim \mathrm{H}^0(C \cap H, \mathcal{O}_{C\cap H}(m)) = \\ = \dim \mathrm{H}^0(C, \mathcal{O}_C(m)) - \dim \mathrm{H}^0(C, \mathcal{O}_C(m-1)) = d \tag{1}$$

by the additivity of the Euler characteristic, Serre vanishing [Har77, Chapter 3, Theorem 5.2], and Lemma 2.3 above. □

Theorem 3.3. *Suppose that $C \subset \mathbb{P}^n$ is a geometrically irreducible curve of degree $d = \deg(C)$ and genus g. Then one has the vanishing*

$$\mathrm{H}^1(C, \mathcal{O}_C(\epsilon)) = 0$$

for all $\epsilon \geq d-2$. If C is geometrically integral then there's also an inequality $g \leq d^2 - 2d + 1$.

Proof. Since these quantities don't change when making an extension of the base field k, it suffices to assume that k is algebraically closed. The proof is devoted to showing that $\mathrm{H}^1(C, \mathcal{O}_C(\epsilon)) = 0$, where $\epsilon \geq d-2$ and $d = \deg(C)$. Indeed, if this were the case then one would have

$$\chi(C, \mathcal{O}_C(d-2)) = \dim \mathrm{H}^0(C, \mathcal{O}_C(d-2)) = d(d-2) + 1 - g \geq 0$$

by Lemma 2.3 above. Rearranging this inequality shows $g \leq d(d-2)+1 = d^2 - 2d + 1$ as claimed.

To start, we let H be a hyperplane constructed as in the proof of Lemma 3.2 and we consider the exact sequence (S1) twisted by $\mathcal{O}_C(m)$ for some $m \geq 0$,

$$0 \to \mathcal{O}_C(m-1) \to \mathcal{O}_C(m) \to \mathcal{O}_{C\cap H}(m) \to 0.$$

By taking cohomology one gets a long exact sequence

$$0 \to \mathrm{H}^0(C, \mathcal{O}_C(m-1)) \to \mathrm{H}^0(C, \mathcal{O}_C(m)) \to \mathrm{H}^0(C \cap H, \mathcal{O}_{C\cap H}(m)) \to \cdots \\ \cdots \to \mathrm{H}^1(C, \mathcal{O}_C(m-1)) \to \mathrm{H}^1(C, \mathcal{O}_C(m)) \to 0. \tag{S2}$$

Note that, in order to show the cohomology $\mathrm{H}^1(C, \mathcal{O}_C(d-2))$ vanishes, it suffices by Serre vanishing [Har77, Chapter 3, Theorem 5.2] to show that the morphism

$$\mathrm{H}^0(C, \mathcal{O}_C(m)) \to \mathrm{H}^0(C \cap H, \mathcal{O}_{C\cap H}(m))$$

of (S2) is surjective for all $m \geq d-1$. Since the following diagram commutes, where every arrow is induced by the corresponding inclusion of varieties,

$$\begin{array}{ccc} \mathcal{O}_C(m) & \longrightarrow & \mathcal{O}_{C\cap H}(m) \\ \uparrow & \nearrow & \\ \mathcal{O}_{\mathbb{P}^n}(m) & & \end{array} \tag{D1}$$

it suffices to show that the morphism, obtained by taking global sections of (D1),

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to \mathrm{H}^0(C \cap H, \mathcal{O}_{C\cap H}(m))$$

is a surjection for all $m \geq d-1$. But this is exactly the content of Lemma 3.1 applied to the intersection $C \cap H$. □

4. The vanishing theorem

In this section we prove a vanishing theorem for the top cohomology of a geometrically integral projective variety $X \subset \mathbb{P}^n$ of dimension r and degree $d = \deg(X) \leq r+1$.

Theorem 4.1. *Suppose that $X \subset \mathbb{P}^n$ is a geometrically integral variety of dimension $r \geq 1$ and of degree $d = \deg(X)$. Then one has the vanishing*

$$\mathrm{H}^r(X, \mathcal{O}_X(\epsilon)) = 0$$

for all $\epsilon \geq d - r - 1$.

Proof. The proof works by induction on the dimension of X. By Theorem 3.3, the claim is true for all geometrically integral curves C where the dimension is $r = 1$. Suppose then that the claim also holds for all geometrically integral varieties $Y \subset \mathbb{P}^n$ of dimension $s < r$ and let $X \subset \mathbb{P}^n$ be a fixed geometrically integral variety of dimension r.

By Bertini's theorem [Jou83, Théorèm 6.10 et Corollaire 6.11], there is a hyperplane $H \subset \mathbb{P}^n$ with intersection $Y = X \cap H$ a geometrically integral variety of dimension $r-1$. Note that, as in Lemma 3.2, there is a short exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{S3}$$

which shows that $\deg(Y) = d$ as well. Twisting (S3) by $\mathcal{O}_Y(m)$ for some $m \geq 0$ and applying cohomology to the sequence (S3) yields the exact sequence

$$\mathrm{H}^{r-1}(Y, \mathcal{O}_Y(m)) \to \mathrm{H}^r(X, \mathcal{O}_X(m-1)) \to \mathrm{H}^r(X, \mathcal{O}_X(m)) \to 0. \tag{S4}$$

By the induction hypothesis, the leftmost cohomology group of (S4) vanishes for

$$m \geq \epsilon + 1 = d - (r-1) - 1 = d - r.$$

By Serre vanishing [Har77, Chapter 3, Theorem 5.2] this means that

$$\mathrm{H}^r(X, \mathcal{O}_X(\epsilon)) = \mathrm{H}^r(X, \mathcal{O}_X(\epsilon+1)) = \cdots = \mathrm{H}^r(X, \mathcal{O}_X(\epsilon+\ell)) = 0$$

as desired. □

Corollary 4.2. *Suppose that $X \subset \mathbb{P}^n$ is a smooth and geometrically connected variety of dimension $r \geq 1$ and of degree $\deg(X) \leq r+1$. Then $\omega_X \neq \mathcal{O}_X$.*

Proof. By Theorem 4.1, the cohomology $\mathrm{H}^r(X, \mathcal{O}_X)$ vanishes. Because of Serre duality [Har77, Chatper 3, Theorem 7.6], the cohomology $\mathrm{H}^0(X, \omega_X)$ also vanishes but, $\mathrm{H}^0(X, \mathcal{O}_X) \neq 0$. □

Example 4.3. Over an algebraically closed field, we can describe all varieties X of dimensions $r = 1, 2$ satisfying the conditions of Corollary 4.2 and which have the additional property that they are nondegenerate in their embedding $X \subset \mathbb{P}^n$. For curves (with $r = 1$), it follows from [EH87, Proposition 0] that any such variety is either $\mathbb{P}^1$ or a smooth conic in $\mathbb{P}^2$. Similarly, by [EH87, Proposition 0] all such surfaces ($r = 2$) are either $\mathbb{P}^2$, a smooth quadric in $\mathbb{P}^3$, a smooth cubic in $\mathbb{P}^3$, or (by [EH87, Theorem 1]) a rational normal cubic scroll in $\mathbb{P}^4$.

References

[EH87] D. Eisenbud and J. Harris, *On varieties of minimal degree (a centennial account)*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 3–13. MR 927946

[GLP83] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), no. 3, 491–506. MR 704401

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Har81] J. Harris, *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 1, 35–68. MR 616900

[Har94] R. Hartshorne, *The genus of space curves*, Ann. Univ. Ferrara Sez. VII (N.S.) **40** (1994), 207–223 (1996). MR 1399631

[Jou83] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983. MR 725671

[Poo04] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127. MR 2144974

[Sta19] The Stacks project authors, *The stacks project*, https://stacks.math.columbia.edu, 2019.

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