# THE ARITHMETIC GENUS OF A COMPLETE INTERSECTION CURVE

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Abstract. The purpose of this short note is to relate two formulas for the genus of a curve that can be realized as a complete intersection in some projective space.

Fix a field $k$. Without any loss of generality, one can suppose that $k$ is algebraically closed throughout this note. Let $X$ be a projective $k$-variety and choose an embedding

$$X \subset \mathbb{P}^n = \mathrm{Proj}(k[x_0, ..., x_n]).$$

We say that $X$ is a *complete intersection* (with respect to this embedding) if $X$ is the vanishing locus $X = V_+(f_1, ..., f_c)$ of $c = \mathrm{codim}(X, \mathbb{P}^n)$ homogeneous equations $f_1, ..., f_c$ of the coordinate ring $k[x_0, ..., x_n]$ that form a regular sequence for this ring.

When $X$ is a complete intersection curve (i.e. $\dim(X) = 1$), the arithmetic genus of $X$ has been calculated in [AS98, Corollary 2].

**Theorem 0.1.** *Suppose that $X = V_+(f_1, ..., f_{n-1}) \subset \mathbb{P}^n$ is a complete intersection curve. Then the arithmetic genus $g(X)$ of $X$ equals*

$$g(X) = \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \sum_{1 \leq a_1 < \cdots < a_i \leq n-1} \binom{d_{a_1} + \cdots + d_{a_i} - 1}{d_{a_1} + \cdots + d_{a_i} - n - 1} \right) \tag{no.1}$$

*where for each $1 \leq i \leq n-1$ we write $d_i = \deg(f_i)$.* □

Briefly, the proof of Theorem 0.1 utilizes the fact that the Koszul complex gives a resolution for the structure sheaf of $X$ by sums of twists of the tautological bundle on $\mathbb{P}^n$; the Euler characteristic of $X$ (and hence the arithmetic genus) can then be determined explicitly from the computation [Sta20, Tag 01XT] of the cohomology of these twists.

The purpose of this note is to prove the following simplification of formula (no.1).

**Theorem 0.2.** *Suppose that $X = V_+(f_1, ..., f_{n-1}) \subset \mathbb{P}^n$ is a complete intersection curve. Then the arithmetic genus $g(X)$ of $X$ equals*

$$g(X) = 1 + \frac{1}{2} (d_1 + \cdots + d_{n-1} - n - 1) \, d_1 \cdots d_{n-1} \tag{no.2}$$

*where for each $1 \leq i \leq n-1$ we write $d_i = \deg(f_i)$.*

**Remark 0.3.** If $X = H_1 \cap \cdots \cap H_{n-1}$ is the intersection of hypersurfaces $H_i \subset \mathbb{P}^n$ such that the sequence

$$H_1, \quad H_1 \cap H_2, \quad H_1 \cap H_2 \cap H_3, \quad \ldots, \quad H_1 \cap \cdots \cap H_{n-1}$$

consists of smooth schemes, then Theorem 0.2 can be proved using the adjunction formula and induction; note that $X$ is not assumed smooth, or even reduced, in Theorem 0.2.

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Before giving the proof, we make some initial observations. Consider the following *set* of points $S^{n-1}_{>0} \subset \mathbb{A}^{n-1}_{\mathbb{Q}}(\mathbb{Z})$ consisting of tuples of integers with positive coordinates

$$S^{n-1}_{>0} = \{(d_1, ..., d_{n-1}) : d_i \in \mathbb{Z}, \quad d_1, ..., d_{n-1} > 0\}. \tag{S0}$$

The arithmetic genus $g(X)$ from (no.2) agrees with the polynomial of $\mathbb{Q}[X_1, ..., X_{n-1}]$

$$g_n(X_1, ..., X_{n-1}) := \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \frac{1}{n!} \sum_{a_1 < \cdots < a_i} \prod_{j=1}^{n} (X_{a_1} + \cdots + X_{a_i} - j) \right) \tag{gn}$$

evaluated at the corresponding point of $S^{n-1}_{>0}$. Because of the following lemma, we'll often work with the latter description of the arithmetic genus.

**Lemma 0.4.** *Fix an integer $n \geq 2$. Let $V \subset \mathbb{A}^{n-1}_{\mathbb{Q}}$ be an arbitrary closed subvariety. Then there is a containment $S^{n-1}_{>0} \subset V$ if and only if $V = \mathbb{A}^{n-1}_{\mathbb{Q}}$. In particular, if a polynomial $f(X_1, ..., X_{n-1}) \in \mathbb{Q}[X_1, ..., X_{n-1}]$ vanishes on $S^{n-1}_{>0}$, then $f(X_1, ..., X_{n-1}) = 0$.*

*Proof.* Let $V = V(f_1, ..., f_m)$ be the affine variety defined as the vanishing locus of some nonconstant polynomials $f_1, ..., f_m \in \mathbb{Q}[X_1, ..., X_n]$. We'll show that there is a point of $S^{n-1}_{>0}$ not contained in $V$; to do this it suffices to work with any of the hypersurfaces $V(f_i)$, and without loss of any generality, we'll assume $V = V(f)$. Since $\mathbb{Q}$ is infinite, there is a point $p \in \mathbb{A}^{n-1}_{\mathbb{Q}}(\mathbb{Q})$ outside of $V$; we can also assume that $p$ has all positive coordinates. Let $\ell$ be the line connecting $p$ and the origin. Then the restriction of $f$ to $\ell$ has finitely many zeros and $\ell$ intersects $S^{n-1}_{>0}$ infinitely often. □

**Lemma 0.5.** *Let $n \geq 3$ be an integer. Then $g_n(1, X_2, ..., X_{n-1}) = g_{n-1}(X_2, ..., X_{n-1})$.*

*Proof.* Identify $S^{n-1}_{>0}$ with the intersection $S^n_{>0} \cap V(X_1 - 1) \subset \mathbb{A}^n_{\mathbb{Q}}$, i.e. with the restriction of $S^n_{>0}$ to the hyperplane where $X_1 = 1$. In this case, $g_n(1, X_2, ..., X_{n-1}) - g_{n-1}(X_2, ..., X_{n-1})$ vanishes on every point of $S^{n-1}_{>0}$, as they both compute the arithmetic genus. Applying lemma 0.4 gives the result. □

**Lemma 0.6.** *Keep notation as in Lemma 0.7. Then there is an equality*

$$g_n(X_1 + 1, X_2, ..., X_{n-1}) = g_n(X_1, X_2, ..., X_{n-1}) + \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \frac{1}{(n-1)!} \sum_{1 < a_2 < \cdots < a_i} \prod_{j=1}^{n-1} (X_1 + \cdots + X_{a_i} - j) \right)$$

*as elements of $\mathbb{Q}[X_1, ..., X_{n-1}]$.*

*Proof.* Restricted to the set $S^{n-1}_{>0}$ of (S0), the polynomial $g_n(X_1, ..., X_{n-1})$ agrees with the function

$$g'_n(X_1, .., X_{n-1}) := \sum_{i=1}^{n-1} (-1)^{i+n-1} \left( \sum_{1 \leq a_1 < \cdots < a_i \leq n-1} \binom{X_{a_1} + \cdots + X_{a_i} - 1}{X_{a_1} + \cdots + X_{a_i} - n - 1} \right).$$

Because of the recursive formula for binomial coefficients,

$$\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$$

the function $g_n'(X_1, ..., X_{n-1})$ satisfies the equality

$$g_n'(d_1+1, d_2, ..., d_{n-1}) = g_n'(d_1, d_2, ..., d_{n-1}) + \sum_{i=1}^{n-1}(-1)^{i+n-1}\left(\sum_{1<a_2<\cdots<a_i}\binom{d_1+\cdots+d_{a_i}-1}{d_1+\cdots+d_{a_i}-n}\right)$$

for any point $(d_1, ..., d_{n-1})$ of $S_{>0}^{n-1}$. In other words, the polynomial

$$g_n(X_1+1, X_2, ..., X_{n-1}) - g_n(X_1, X_2, ..., X_{n-1}) \\ - \sum_{i=1}^{n-1}(-1)^{i+n-1}\left(\frac{1}{(n-1)!}\sum_{1<a_2<\cdots<a_i}\prod_{j=1}^{n-1}(X_1+\cdots+X_{a_i}-j)\right)$$

vanishes restricted to $S_{>0}^{n-1}$; the claim follows from Lemma 0.4. □

The proof of Theorem 0.2 is dependent on the following lemma.

**Lemma 0.7.** *For any $n \geq 2$, there's an equality*

$$g_n(X_1, ..., X_{n-1}) = 1 + X_1 \cdots X_{n-1} h_n(X_1, ..., X_{n-1})$$

*for some polynomial $h_n(X_1, ..., X_{n-1}) \in \mathbb{Q}[X_1, ..., X_{n-1}]$ with*

$$h_n(X_1, ..., X_{n-1}) = a_1 X_1 + \cdots + a_{n-1} X_{n-1} + c$$

*for some $a_1, ..., a_{n-1}, c \in \mathbb{Q}$.*

*Proof.* The claim is clear when $n = 2$ so assume $n \geq 3$. We'll use the recursive formula

$$g_n(X_1+1, X_2, ..., X_{n-1}) = g_n(X_1, X_2, ..., X_{n-1}) \\ + \sum_{i=1}^{n-1}(-1)^{i+n-1}\left(\frac{1}{(n-1)!}\sum_{1<a_2<\cdots<a_i}\prod_{j=1}^{n-1}(X_1+\cdots+X_{a_i}-j)\right).$$

After setting $X_1 = 0$ in the above recursion one gets the equality

$$g_n(1, X_2, ..., X_{n-1}) = g_n(0, X_2, ..., X_{n-1}) - 1 + g_{n-1}(X_2, ..., X_{n-1}).$$

Since there's also an equality $g_n(1, X_2, ..., X_{n-1}) = g_{n-1}(X_2, ..., X_{n-1})$ by Lemma 0.5, it follows that

$$g_n(0, X_2, ..., X_{n-1}) - 1 = 0.$$

As $g_n(X_1, ..., X_{n-1})$ is symmetric in the variables $X_i$, it follows $X_i$ divides $g_n(X_1, ..., X_{n-1}) - 1$ for each $1 \leq i \leq n-1$, which proves the first part of the lemma that there's an equality

$$g_n(X_1, ..., X_{n-1}) = 1 + X_1 \cdots X_{n-1} h_n(X_1, ..., X_{n-1})$$

for some polynomial $h_n(X_1, ..., X_{n-1}) \in \mathbb{Q}[X_1, ..., X_{n-1}]$.

Now we show that $h_n(d_1, ..., d_{n-1})$ as defined above is linear of the given form. To do this, we work with the individual summands

$$\frac{1}{n!}\prod_{j=1}^{n}(X_{a_1}+\cdots+X_{a_i}-j). \tag{FF}$$

Subtracting 1 from $g_n(X_1, ..., X_{n-1})$ and dividing the result by $X_1 \cdots X_{n-1}$ is a polynomial in $\mathbb{Q}[X_1, ..., X_{n-1}]$ so, after expanding any of the summands (FF) and dividing by $X_1 \cdots X_{n-1}$, all monomials with nontrivial denominator must vanish after summing over all other terms with the same denominator. This leaves just the last term of the sum from (gn), when $i = n-1$, as a contributing factor to $h_n(X_1, ..., X_{n-1})$. Expanding this term shows

$$\frac{1}{n!}\prod_{j=1}^{n}(X_1 + \cdots + X_{n-1} - j) =$$
$$\frac{1}{n!}\left((X_1 + \cdots + X_{n-1})^n + (-1)^{n-1}\frac{n(n+1)}{2}(X_1 + \cdots + X_{n-1})^{n-1} + L(X_1, ..., X_{n-1})\right)$$

where the summand $L(X_1, ..., X_{n-1})$ is comprised of terms of degree smaller than $n-1$, and doesn't contribute to the polynomial $h_n(X_1, ..., X_{n-1})$. After expanding $(X_1 + \cdots + X_{n-1})^n$, the monomial summands divisible by $X_1 \cdots X_{n-1}$ are multiples of $X_i(X_1 \cdots X_{n-1})$ for varying $1 \leq i \leq n$; after expanding $(X_1 + \cdots + X_{n_1})^{n-1}$, the monomial summands divisible by $X_1 \cdots X_{n-1}$ are multiples of $X_1 \cdots X_{n-1}$. As $h_n(X_1, ..., X_{n-1})$ is the polynomial that one gets after dividing the sum of these summands by $X_1 \cdots X_{n-1}$, this shows $h_n(X_1, ..., X_{n-1})$ is linear of the given form, as claimed. ☐

*Proof of Theorem 0.2.* By Lemma 0.7, we have that

$$g_n(d_1, ..., d_{n-1}) = 1 + d_1 \cdots d_{n-1} h_n(d_1, ..., d_{n-1})$$

for a linear polynomial

$$h_n(d_1, ..., d_{n-1}) = a_1 d_1 + \cdots + a_{n-1} d_{n-1} + c.$$

Note that, when $n = 2$, the equation (no.1) becomes

$$g_2(d_1) = \binom{d_1 - 1}{d_1 - 3} = \frac{(d_1 - 1)(d_1 - 2)}{2} = 1 + \frac{1}{2}(d_1 - 3)d_1.$$

Hence, when $n \geq 3$, one finds

$$g_2(d_i) = g_n(1, ..., d_i, ..., 1) = 1 + d_i h_n(1, ..., d_i, ..., 1)$$

by setting $d_j = 1$ for all $j \neq i$. It follows that $a_i = 1/2$ for all $1 \leq i \leq n-1$. Finally, one can solve for $c$ using the relation $0 = g_n(1, ..., 1)$ where $d_i = 1$ for all $1 \leq i \leq n-1$. ☐

## References


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