# RATIONAL EQUIVALENCE OF CURVES ON A SEVERI–BRAUER VARIETY

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Abstract. Let $A$ be a central simple algebra over a field $F$ and of odd index. We show that the torsion subgroup of the Chow group $\mathrm{CH}_1(X)$ of 1-cycles on the Severi–Brauer variety $X = \mathbf{SB}(A)$ is trivial if and only if any curve in $X$ is rationally equivalent to a member of some collection of birationally equivalent curves on $X$.

## 1. Introduction

Chow groups of Severi–Brauer varieties have been of interest since they first appeared in the proof of the Merkurjev–Suslin norm residue isomorphism theorem [MS82]. In the proof of this theorem, the Chow groups of Severi–Brauer varieties associated to central division algebras of prime degree are shown to be torsion free.

Suslin later conjectured that the Chow groups of any Severi–Brauer variety were torsion free [Sus84, Remark 10.14 and Conjecture 24.6]. The first counterexample to this conjecture, a Severi–Brauer variety whose Chow ring contained nontrivial torsion, appeared in [Mer95]. Since then there have been a number of constructions of nontrivial torsion in the Chow groups of specific Severi–Brauer varieties [Kar95, Kar98, Kar17, Bae15, KM19, Mac20c].

All known examples of nontrivial torsion in the Chow groups of a Severi–Brauer variety occur in the Chow groups of cycles of reasonably large dimension. In this text, we study Chow groups of dimension one cycles on a Severi–Brauer variety. This is the smallest dimension where the structure of these Chow groups is unknown since the Chow group of cycles of dimension zero is torsion free [CM06, Kra10].

The main results of this text are the Corollaries 2.3 and 3.8. Together these corollaries prove that on any Severi–Brauer variety associated to a central simple algebra of odd index there is a collection of birationally equivalent curves so that the Chow group of cycles of dimension one is torsion free if and only if the rational equivalence class of an arbitrary

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curve is represented by a member of this collection.

**Notation**. We use the following notation throughout:

- $F$ is a base field
- $A$ is a simple $F$-algebra with center $F$ and finite $F$-dimension
- $D_A$ is the underlying division algebra of $A$
- the degree of $A$ is the number $\deg(A) = \sqrt{\dim_F(A)}$
- the index of $A$ is the number $\mathrm{ind}(A) = \sqrt{\dim_F(D_A)}$
- $X = \mathbf{SB}(A)$ is the Severi–Brauer variety of dimension $\deg(A) - 1$.

**Conventions**. We use the following conventions throughout:

- a variety is an integral scheme that is separated and of finite type over a base field
- a curve is a scheme of dimension one that is separated and of finite type over a base field.

## 2. Chow groups

Let $A$ be a simple $F$-algebra with center $F$ and finite $F$-dimension. Associated to $A$ is the Severi–Brauer variety $X = \mathbf{SB}(A)$. By definition $X$ is the subvariety of the Grassmannian $\mathbf{Gr}(\deg(A), A)$ whose $R$-points $X(R) \subset \mathbf{Gr}(\deg(A), A)(R)$, for any finite type $F$-algebra $R$, correspond exactly to the projective summands of $A \otimes_F R$ that are also minimal right ideals of $A \otimes_F R$. In this section, we make some observations regarding the Chow groups $\mathrm{CH}_i(X)$ of dimension-$i$ cycles on $X$.

Recall that there is a canonical vector bundle $\zeta_X$ on $X$ coming from the pullback along the embedding $X \subset \mathbf{Gr}(\deg(A), A)$ of the universal bundle of dimension-$\deg(A)$ $F$-subspaces of $A$. The fiber of $\zeta_X$ over an $R$-point $x$ of $X(R)$ is the right ideal corresponding to $x$. This allows one to define vector bundles

$$\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$$

for every $i \geq 0$ and for some choice of simple left $A^{\otimes i}$-module $M_i$. Conventionally, we set $\zeta_X(i) := \zeta_X(-i)^\vee$ if $i < 0$.

We write $\mathrm{CT}(1; X)$ for the subring of $\mathrm{CH}(X)$ generated by the Chern classes of $\zeta_X(1)$. The gradings on $\mathrm{CH}(X)$ induces gradings on $\mathrm{CT}(1; X)$. We write

$$\mathrm{CT}^i(X) = \mathrm{CT}(1; X) \cap \mathrm{CH}^i(X) \quad \text{and} \quad \mathrm{CT}_i(X) = \mathrm{CT}(1; X) \cap \mathrm{CH}_i(X)$$

for the corresponding subgroups. By [KM19, Proposition A.8], there is an identification $\mathrm{CT}_i(X) = \mathbb{Z}$.

**Lemma 2.1.** *For any integer* $0 \leq i \leq \dim(X)$*, there exists a closed subvariety* $V \subset X$ *with* $\dim(V) = i$ *so that the rational equivalence class* $[V]$ *in* $\mathrm{CH}_i(X)$ *is contained in* $\mathrm{CT}_i(X)$*.*

*Proof.* If $A$ is split, then $X = \mathbb{P}^n$ for some $n \geq 0$ and $\zeta_X(1) = \mathcal{O}_{\mathbb{P}^n}(-1)$. In this case the claim is immediate so that we can, from now on, assume that $A$ is nontrivial. In particular, we can assume that the base field $F$ is infinite.

The line bundle $\zeta_X(-\mathrm{ind}(A)) = \det(\zeta_X(1)^\vee)$ is very ample so there is a closed immersion

$$\rho : X \to \mathbb{P}(W)$$

with $W = \mathrm{H}^0(X, \zeta_X(-\mathrm{ind}(A)))$. For each integer $0 \leq i \leq \dim(X)$, Bertini's theorem [Jou83, Théorème 6.10 et Corollaire 6.11] then gives a linear subspace $H_i \subset \mathbb{P}(W)$ so that:

(1) the intersection $V_i = H_i \cap \rho(X)$ is smooth for all $i \geq 0$,
(2) the intersection $V_i$ is geometrically integral for all $i > 0$,
(3) and there is an equality $\mathrm{codim}_{\mathbb{P}(W)}(H_i) = \mathrm{codim}_X(V_i)$.

The diagram below, depicting the situation, is Cartesian for all $i \geq 0$.

$$\begin{array}{ccc} V_i = H_i \cap \rho(X) & \xrightarrow{\rho|_{V_i}} & H_i \\ \downarrow{\scriptstyle \varphi_i|_{V_i}} & & \downarrow{\scriptstyle \varphi_i} \\ X & \xrightarrow{\rho} & \mathbb{P}(W). \end{array}$$

By [EKM08, Corollary 55.4] there is an equality of maps

$$(\varphi_i|_{V_i})_* \circ (\rho|_{V_i})^* = \rho^* \circ \varphi_{i*} : \mathrm{CH}^0(H_i) \to \mathrm{CH}_i(X).$$

By [EKM08, Proposition 55.6], one has $(\rho|_{V_i})^*([H_i]) = [V_i]$ so that

$$\rho^*([H_i]) = \rho^* \circ \varphi_{i*}([H_i]) = (\varphi_i|_{V_i})_* \circ (\rho|_{V_i})^*([H_i]) = (\varphi_i|_{V_i})_*([V_i]) = [V_i].$$

Since the pullback respects Chern classes, there is an integer $m_i$ so that

$$[V_i] = \rho^*([H_i]) = \rho^*(c_1(\mathcal{O}_{\mathbb{P}(W)}(1))^{m_i}) = c_1(\zeta_X(-\mathrm{ind}(A)))^{m_i}$$

is contained in $\mathrm{CT}_i(X)$ as desired. □

We now turn to consider the quotients

$$\mathrm{Q}^i(X) = \mathrm{CH}^i(X)/\mathrm{CT}^i(X) \quad \text{and} \quad \mathrm{Q}_i(X) = \mathrm{CH}_i(X)/\mathrm{CT}_i(X).$$

From the short exact sequence

$$0 \to \mathbb{Z} = \mathrm{CT}_i(X) \to \mathrm{CH}_i(X) \to \mathrm{Q}_i(X) \to 0$$

it follows that the groups $\mathrm{Q}_i(X)$ are torsion and there is an inclusion

$$0 \to \mathrm{Tor}_1(\mathrm{CH}_i(X), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tor}_1(\mathrm{Q}_i(X), \mathbb{Q}/\mathbb{Z}).$$

The following proposition is the crux of this text.

**Proposition 2.2.** *Let $V \subset X$ be any closed and irreducible subscheme with $\dim(V) = i$. If the pushforward along the first projection*

$$\mathrm{CH}_i(X \times W) \to \mathrm{CH}_i(X)$$

*has image contained in $\mathrm{CT}_i(X)$ for any closed subscheme $W \subsetneq V$, then there is an equality $[V] = [V']$ in $\mathrm{Q}_i(X)$ for any subscheme $V' \subset X$ birationally equivalent to $V$.*

*Proof.* As $V$ and $V'$ are birationally equivalent, there are dense opens $U \subset V$, $U' \subset V'$, and an isomorphism $f : U' \to U$. We write

$$\Delta_V \subset V \times V \subset X \times V \quad \text{and} \quad \overline{\Gamma}_f \subset V' \times V \subset X \times V$$

for the diagonal and for the closure of the graph of $f$ respectively.

Consider the following diagram.

$$\begin{array}{ccccccc} \varinjlim \mathrm{CH}_i(X \times W) & \longrightarrow & \mathrm{CH}_i(X \times V) & \longrightarrow & \mathrm{CH}_0(X_{F(V)}) & \longrightarrow & 0 \\ & \searrow^{\varinjlim \pi|_{X \times W *}} & \downarrow{\scriptstyle \pi_*} & & & & \\ & & \mathrm{CH}_i(X) & & & & \end{array}$$

The top row is the colimit of the exact localization sequences with respect to all open subschemes $X \times (V \setminus W) \subset X \times V$. The vertical arrow is the pushforward along the projection $\pi : X \times V \to X$ and the diagonal arrow is the colimit of the pushforwards along the projections $\pi|_{X \times W} : X \times W \to X$ as $W$ varies over all closed subschemes $W \subsetneq V$.

Since $X_{F(V)}$ has an $F(V)$-rational point, the group $\mathrm{CH}_0(X_{F(V)}) = \mathbb{Z}$ is infinite cyclic with generator the class of a rational point. Since both $[\Delta_V]$ and $[\overline{\Gamma}_f]$ restrict to the class of a rational point in $\mathrm{CH}_0(X_{F(V)})$, it follows that there is a subscheme $W \subset V$ and an element $\phi$ of $\mathrm{CH}_i(X \times W)$ so that

$$\pi|_{X \times W *}(\phi) = \pi_*([\Delta_V] - [\overline{\Gamma}_f]) = [V] - [V'].$$

By assumption, the left side of this equation is contained in $\mathrm{CT}_i(X)$, so that $[V] = [V']$ in $\mathrm{Q}_i(X)$ as claimed. □

Let $\mathrm{CH}_1^{\pm}(X) = \mathrm{CH}_1(X)/\sim_{\pm}$ where $\sim_{\pm}$ is the equivalence relation on the set $\mathrm{CH}_1(X)$ identifying a class with its opposite, i.e. $\tau \sim_{\pm} -\tau$.

**Corollary 2.3.** *Suppose that there exists a collection of curves $\{\mathscr{C}_\tau\}_\tau$ on $X$, indexed over the elements $\tau \in \mathrm{CH}_1^{\pm}(X) \setminus \{0\}$, satisfying:*

1. *the class $[\mathscr{C}_\tau] = \tau$ in $\mathrm{CH}_1^{\pm}(X)$*
2. *for all classes $\sigma, \tau \in \mathrm{CH}_1^{\pm}(X) \setminus \{0\}$, the curve $\mathscr{C}_\tau$ is birationally equivalent with $\mathscr{C}_\sigma$.*

*Then $\mathrm{CH}_1(X) = \mathbb{Z}$ is torsion free.*

*Proof.* The assumptions of Proposition 2.2 hold for $V \subset X$ any curve by [Mac20b, Lemma 3.5]. Hence (2) implies that every curve $\mathscr{C}_\tau$ represents the same class in $\mathrm{Q}_1(X)$. It follows from (1) that every curve contained in $X$ has the same class in $\mathrm{Q}_1(X)$. Since there is a curve $D \subset X$ whose rational equivalence class $[D]$ is contained in $\mathrm{CT}_1(X)$ by Lemma 2.1, we find that $\mathrm{Q}_1(X) = 0$ and $\mathrm{CH}_1(X) = \mathbb{Z}$. □

## 3. Constructing curves

Let $K \subset D_A$ be a separable maximal subfield of the division algebra underlying $A$. Let $E$ be the Galois closure of $K/F$ in some separable closure of $F$ and write $G = \mathrm{Gal}(E/F)$ for the Galois group. The variety $X = \mathbf{SB}(A)$ is a form of $\mathbb{P}^n$ twisted along a cocycle representing the class of $A$ or $X$ in $\mathrm{H}^1(G, \mathrm{PGL}_n)$ with $n = \deg(A)$. In this section, we use descent along this cocycle to construct a collection of curves in $X$.

In the split case $X = \mathbb{P}^n$ and $n \geq 2$, the Chow group $\mathrm{CH}_1(X) = \mathbb{Z}$ is infinite cyclic with generator the rational equivalence class of any one dimensional linear subspace $L \subset X$. The *degree* of a curve $C \subset X$ is the integer $\deg(C)$ so that $[C] = \deg(C)[L]$ in $\mathrm{CH}_1(X)$.

**Lemma 3.1.** *Let $X = \mathbb{P}^n$ with $n \geq 2$. Then for each integer $d \geq 1$ one can choose a curve $\mathscr{C}_d \subset X$ so that $\mathscr{C}_d(F) \neq \emptyset$ and:*

(1) *the degree of the curve $\mathscr{C}_d$ is $\deg(\mathscr{C}_d) = d$,*
(2) *and $\mathscr{C}_d$ is birationally equivalent with $\mathscr{C}_e$ for all $d, e \geq 1$.*

*Proof.* If $d = 1$, we take $\mathscr{C}_d = L$. If $d > 1$, consider the composition

$$\mathbb{P}^1 \to \mathbb{P}^d \dashrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^n$$

of the $d$th Veronese embedding, a projection inducing a birational equivalence of this Veronese curve with its image, and then a linear inclusion to $\mathbb{P}^n$. Taking $\mathscr{C}_d$ to be the image of this composition we get a collection of curves with the desired properties. □

In the general case $X = \mathbf{SB}(A)$, we call the *degree* of a curve $C \subset X$ the integer $\deg(C)$ so that $[C_K] = \deg(C)[L]$ in $\mathrm{CH}_1(X_K)$ for any one dimensional linear subspace $L \subset X_K$. This definition is independent of the field $K$. The following theorem is due to Karpenko.

**Theorem 3.2** ([Mac20a, Corollary 3.8]). *Let $X = \mathbf{SB}(A)$ be as above. Assume additionally $\mathrm{ind}(A)$ is odd. Let $K \supset F$ be a splitting field for $X$ and write $\pi_K : X_K \to X$ for the projection map. Then the pullback gives an identification*

$$\pi_K^* \mathrm{CH}_1(X) = \mathrm{ind}(A)\mathbb{Z} \subset \mathbb{Z} = \mathrm{CH}_1(X_K).$$

*In particular, every curve $C \subset X$ has degree a multiple of $\mathrm{ind}(A)$.*

**Remark 3.3.** If $\text{ind}(A)$ is even, then it may not be true that $\text{ind}(A)$ divides the degree $\deg(C)$ of any curve $C \subset X = \mathbf{SB}(A)$. Some explicit examples are given in [Kar96, Theorem 2.5] and [Kar17, Corollary 3.16].

The main theorem of this section is:

**Theorem 3.4.** *Let $X = \mathbf{SB}(A)$ as above and suppose $\dim(X) \geq 2$. Then for each integer $d \geq 1$, one can find a curve $\mathscr{C}_d \subset X$ so that:*

1. *the degree of the curve $\mathscr{C}_d$ is $\deg(\mathscr{C}_d) = \text{ind}(A)d$,*
2. *and $\mathscr{C}_d$ is birationally equivalent with $\mathscr{C}_e$ for all $d, e \geq 1$.*

The proof of Theorem 3.4 is broken into two lemmas. The idea is to construct, for each $d \geq 1$, a $G$-orbit of isomorphic degree $d$ curves in $X_E$ whose union is defined over $K$. These curves descend to a collection as desired because of the following:

**Lemma 3.5.** *Let $H \subset G$ be a subgroup of $G$ and $H\backslash G$ the set of right cosets of $H$. Suppose that both $\{C_g\}_{g\in H\backslash G}$ and $\{D_g\}_{g\in H\backslash G}$ are $G$-orbits of curves in $X_E$ labeled so that $h(C_g) = C_{hg}$ and $h(D_g) = D_{hg}$ for all $h$ in $G$. Then the unions*

$$\bigcup_{g\in H\backslash G} C_g \quad \text{and} \quad \bigcup_{g\in H\backslash G} D_g$$

*descend to curves $\mathscr{C}$ and $\mathscr{D}$ on $X$ respectively. Moreover, if $C_g$ is birationally equivalent with $D_g$ for any $g$ in $H\backslash G$, then $\mathscr{C}$ is birationally equivalent with $\mathscr{D}$.*

*Proof.* The curves $\mathscr{C}$ and $\mathscr{D}$ exist by descent so we're left to show the last statement. Let $f_g : U_g \to U'_g$ be an isomorphism between an open subset $U_g \subset C_g$ and an open $U'_g \subset D_g$. For any $h$ in $G$, define $f_h : hg^{-1}(U_g) \to hg^{-1}(U'_g)$ by the formula $f_h = (hg^{-1}) \circ f_g \circ (gh^{-1})$. Then the map

$$\tilde{\Phi} = \bigcup_{g\in G} f_g : \bigcup_{g\in H\backslash G} C_g \dashrightarrow \bigcup_{g\in H\backslash G} D_g$$

is $G$-equivariant so it descends to a birational map $\Phi : \mathscr{C} \dashrightarrow \mathscr{D}$. □

There are two different ways to construct these collections of curves, depending on the dimension of $X$, and both are necessary. We assume that $F$ is infinite in the following Lemmas 3.6 and 3.7.

**Lemma 3.6.** *Assume that $\text{ind}(A) = [E : F] = \deg(A)$. Then there exists a point $x$ in $X$ with residue field $F(x) = E$ so that the $E$-points of $x_E$ linearly span $X_E$. Labeling the $E$-points $\{x_g\}_{g\in G}$ of $x_E$ so that $h(x_g) = x_{hg}$ for all $h \in G$, one can find curves $\mathscr{C}_d \subset X_E$ for each integer $d \geq 1$ so that:*

(1) *the degree of the curve* $\mathscr{C}_d$ *is* $\deg(\mathscr{C}_d) = d$,
(2) *the curve* $\mathscr{C}_d$ *passes through only* $x_g$*, i.e.* $\mathscr{C}_d \cap x_E = x_g$,
(3) *and* $\mathscr{C}_d$ *is birationally equivalent with* $\mathscr{C}_e$ *for all* $d, e \geq 1$.

*Proof.* Note that if $x$ is any point in $X$ with $F(x) = E$, then the $E$-points of $x_E$ span a $G$-invariant linear subspace of $X_E$ which must be all of $X_E$ since $A$ is a division algebra.

Now it's possible to find a collection of curves $\mathscr{C}'_d \subset X$ satisfying both (1) and (3) above by Lemma 3.1. To see that we can find a collection of curves $\mathscr{C}_d$ satisfying (1), (2), and (3) we show that, for any fixed $d \geq 1$, there is an automorphism $\alpha$ of $X_E$ so that $\alpha(\mathscr{C}'_d) \cap x_E = x_g$. Labeling $\mathscr{C}_d = \alpha(\mathscr{C}'_d)$ we get the desired collection. For this, we pick an identification $X_E \cong \mathbb{P}^n_E$ and inductively choose $E$-rational points $p_0, ..., p_n$ so that:

(1) $p_0$ is in $\mathscr{C}'_d$
(2) $p_1$ is in $X_E \setminus \mathscr{C}'_d$
(3) $p_2$ is in $X_E \setminus (\mathscr{C}'_d \cup L(p_0, p_1))$ where $L(p_0, p_1)$ is the line in $X_E$ containing both $p_0$ and $p_1$, ...
($n$) and with $p_n$ an $E$-rational point in $X_E \setminus (\mathscr{C}'_d \cup L(p_0, ..., p_{n-1}))$ where $L(p_0, ..., p_{n-1})$ is the linear space in $X_E$ of dimension $n-1$ containing $p_0, ..., p_{n-1}$.

We can then choose a labeling $\{x_i\}_{i=0}^n$ of the $E$-points of $x_E$ so that $x_g = x_0$ and, since $x_E$ spans $X_E$ linearly, there exists an automorphism $\beta$ of $X_E \cong \mathbb{P}^n_E$ defined by

$$\beta(x_0) = p_0, ..., \beta(x_n) = p_n.$$

Taking $\alpha = \beta^{-1}$ completes the proof. □

**Lemma 3.7.** *Suppose that* $\deg(A) \geq 4$*. Let* $x$ *be any point in* $X$ *with residue field* $F(x) = K$*. Let* $H \subset G$ *be the subgroup such that* $K = E^H$*. Label the* $E$*-points* $\{x_g\}_{g \in H\setminus G}$ *of* $x_E$ *so that* $h(x_g) = x_{hg}$ *for all* $h \in G$*. Then there is a plane* $P_g = \mathbb{P}^2_E \subset X_E$ *so that* $P_g \cap x_E = x_g$ *and one can find curves* $\mathscr{C}_d \subset X_E$ *for each integer* $d \geq 1$ *so that:*

(1) *the degree of the curve* $\mathscr{C}_d$ *is* $\deg(\mathscr{C}_d) = d$,
(2) *the curve* $\mathscr{C}_d$ *passes through only* $x_g$*, i.e.* $\mathscr{C}_d \cap x_E = x_g$,
(3) *and* $\mathscr{C}_d$ *is birationally equivalent with* $\mathscr{C}_e$ *for all* $d, e \geq 1$.

*Proof.* Suppose that we can find a plane $P_g$ with the specified property. By Lemma 3.1 we can find, for each integer $d \geq 1$, a curve $\mathscr{C}'_d \subset P_g$ satisfying both (1) and (3). Changing by an automorphism $\alpha$ of $P_g$, we can move any rational point on $\mathscr{C}'_d$ to $x_g$ to get a curve $\mathscr{C}_d = \alpha(\mathscr{C}'_d)$ that now also satisfies (2). So it suffices to prove that $P_g$ exists.

Identify $X_E \cong \mathbb{P}(V)$ for an $E$-vector space $V$ with $\dim(V) \geq 4$. The points $\{x_g\}_{g \in H\backslash G}$ correspond to lines $\{L_g\}_{g \in H\backslash G}$ in $V$. Consider the proper variety

$$W \subset \mathbb{P}(V/L_g) \times \mathbf{Gr}(2, V/L_g)$$

consisting of pairs $(L, P)$ where $L \subset P$. Let $\pi_1$ and $\pi_2$ be the first and second projections from this product respectively. The set of planes $P_g \subset X_E$ with $P_g \cap x_E = x_g$ corresponds to the set of $E$-rational points in the open complement

$$\mathbf{Gr}(2, V/L_g) \setminus \pi_2 \left( W \cap \bigcup_{h \in (H\backslash G)\setminus\{g\}} \pi_1^{-1}(\{L_h\}) \right)$$

which is nonempty because of our assumption $\dim(V) \geq 4$. □

*Proof of Theorem 3.4.* If $\deg(A) = 3$ and $A$ is nonsplit, then there is a Galois field extension $F \subset E$ of degree $[E : F] = 3$ with $A \otimes_F E$ is split by Wedderburn's Theorem [KMRT98, Theorem 19.2]. By Lemma 3.6, for any fixed $g$ in $G = \mathrm{Gal}(E/F)$ and for each integer $d \geq 1$, there are curves $C_g^d \subset X_E$ so that the following hold:

(1) the degree of the curve $C_g^d$ is $\deg(C_g^d) = d$,
(2) the curve $C_g^d$ passes through only $x_g$, i.e. $C_g^d \cap x_E = x_g$,
(3) and $C_g^d$ is birationally equivalent with $C_g^e$ for all $d, e \geq 1$.

Similarly, if $\deg(A) \geq 4$ then using Lemma 3.7 one can find curves $C_g^d \subset X_E$ with $g \in H/G$ satisfying the properties (1), (2), and (3).

In either case, for each $d \geq 1$, the Galois orbit of $C_g^d$ gives a set of curves $\{C_g^d\}_g$ labeled so that $h(C_g^d) = C_{hg}^d$. The union of the curves in this orbit descends to a curve $\mathscr{C}_d \subset X$. The collection of these curves for varying integers $d \geq 1$ has both of the properties:

(1) the degree of $\mathscr{C}_d$ is $\deg(\mathscr{C}_d) = \mathrm{ind}(A)d$,
(2) and $\mathscr{C}_d$ is birationally equivalent with $\mathscr{C}_e$ for all $d, e \geq 1$.

The first property (1) follows from the construction of $\mathscr{C}_d$. The second property (2) follows from Lemma 3.5. □

With Theorem 3.4 proved, we get a converse to Corollary 2.3.

**Corollary 3.8.** *If* $\mathrm{ind}(A)$ *is odd, then* $\mathrm{CH}_1(X)$ *is torsion free only if there exists a collection of curves* $\{\mathscr{C}_\tau\}_\tau$ *on* $X$ *indexed by elements* $\tau \in \mathrm{CH}_1^{\pm}(X) \setminus \{0\}$ *and satisfying the properties:*

(1) *the class* $[\mathscr{C}_\tau] = \tau$ *in* $\mathrm{CH}_1^{\pm}(X)$
(2) *for all classes* $\sigma, \tau \in \mathrm{CH}_1^{\pm}(X) \setminus \{0\}$*, the curve* $\mathscr{C}_\tau$ *is birationally equivalent with* $\mathscr{C}_\sigma$*.*

*Proof.* If $\mathrm{CH}_1(X)$ is torsion free, then by Theorem 3.2 we can identify $\mathrm{CH}_1(X) = \mathbb{Z}$ with a generator being any curve having degree $\mathrm{ind}(A)$. A collection of such curves is then given by Theorem 3.4. ☐

**Remark 3.9.** The group $\mathrm{CH}_1(X)$ is known to be torsion free only in a few cases: if $A$ has almost square-free index [Mer95, Proposition 1.15]; if $(\mathrm{ind}(A), 8) \leq 4$ and $X$ is generic [Mac20a, Theorem 4.1].

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